Algorithms and Data Structures

Sorting beyond Value Comparisons

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Content of this Lecture

- Radix Exchange Sort
  - Sorting bitstrings in linear time (almost)
- Bucket Sort (aka (LSD) Radix Sort)

Knowledge

- Until now, we did not use any knowledge on the nature of the values we sort
  - Strings, integers, reals, names, dates, revenues, person‵s age
  - Only comparison we used: “value1 < value2”
    - Exception: Our (refused) suggestion (max-min)/2 for selecting the pivot element in Quicksort (how can we do this for strings?)
- Use knowledge on data type: Positive integers
- First example
  - Assume a list S of n different integers, $\forall i$: $1 \leq S[i] \leq n$
  - How can we sort S in O(n) time and with only n extra space?

Sorting Permutations

```
S: array_permuted_numbs;
B: array_of_size_|S|
for i:= 1 to |S| do
  B[S[i]] := S[i];
end for;
```

- Very simple
  - If we have all integers [1, n], then the final position of value i must be i
  - Obviously, we need only one scan and only one extra array (B)
- Knowledge we exploited
  - There are n different, unique values
  - The set is „dense“ – no value between 1 and n is missing
  - It follows that the position of a value in the sorted list can be derived from the value

Removing Pre-Requisites

- Assume S is not dense (no duplicates)
  - n different integers each between 1 and m with m>n
  - For a given value S[i], we do not know any more its target position
    - How many values are smaller?
    - At most min(S[i], n)
    - At least max(n-(m-S[i]), 0)
  - This is almost the usual sorting problem, and we cannot do much
    - We can actually sort such a S is O(m) – how?
- Assume S has duplicates
  - S contains n values, where each value is between 1 and m and appears in S with m<n
  - Again: We cannot directly infer the position of S[i] from i alone

Second Example: Sorting Binary Strings

- Assume that all keys are binary strings (bistrings) of equal length
  - E.g., integers in machine representation
- The most important position is the left-most bit, and it can have only two different values
  - Alphabet size is 2 in bitstrings

1110101011010	s_1
1110101011010	s_2
0011010101011	s_3
0101010100101	...
0111010101001	
1110101010110	
1010101111010	
1000001100101	
1010101110110	
1000101101011	s_{10}

Second Example: Sorting Binary Strings

- We can sort all keys by first position with a single scan
  - All values with leading 0 => list B0
  - All values with leading 1 => list B1
  - Requires 2*n additional space
  - But …

1110101011010	S_1
1110101011010	S_2
0011010101011	S_3
0101010100101	S_4
0111010101001	S_5
1110101010110	S_6
1010101111010	S_7
1000001100101	S_8
1010101110110	S_9
1000101101011	S_{10}

⇒

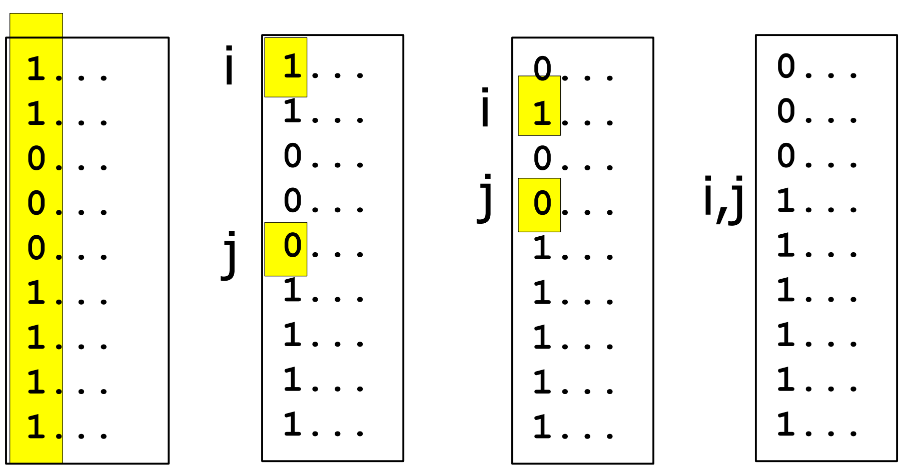
B_0	0011010101011	S_3
	0101010100101	S_4
	0111010101001	S_5
B_1	1110101011010	S_1
	1110101011010	S_2
	1110101010110	S_6
	1010101111010	S_7
	1000001100101	S_8
	1010101110110	S_9
	1000101101011	S_{10}

```
S: array_bitstrings;
B0: array_of_size_|S|
B1: array_of_size_|S|
j0 := 1;
j1 := 1;
for i:= 1 to |S| do
  if S[i][1]=0 then
    B0[j0] := S[i];
    j0 := j0 + 1;
  else
    B1[j1] := S[i];
    j1 := j1 + 1;
  end if;
end for;
return B0[1..j0]+B1[1..j1];
```

Improvement

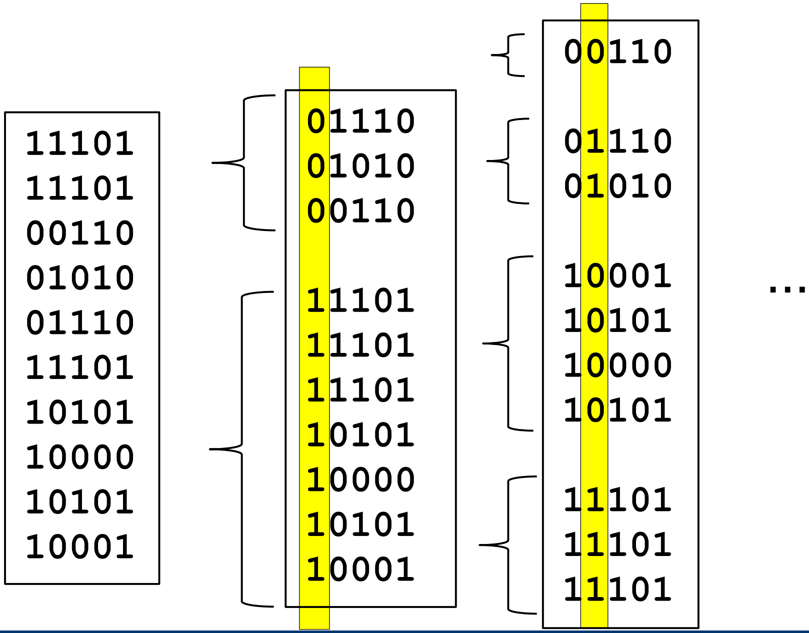
- Recall QuickSort
  - Call **divide*(S,1,1,|S|)**
    - k, l, r, and return value will be used in a minute
  - Note that we return j, the position of the first 1
- O(1) additional space

```
func int divide*(S array;
                     k,l,r: int) {
  i := l;
  j := r;
  repeat
    while S[i][k]=0 and i<j do
      i := i+1;
    end while;
    while S[j][k]=1 and i<j do
      j := j-1;
    end while;
    swap(S[i], S[j]);
  until i=j;
  if S[r][k]=0 then //only zeros
    j:=j+1;
  end if
  return j;   // first "1"
}
```

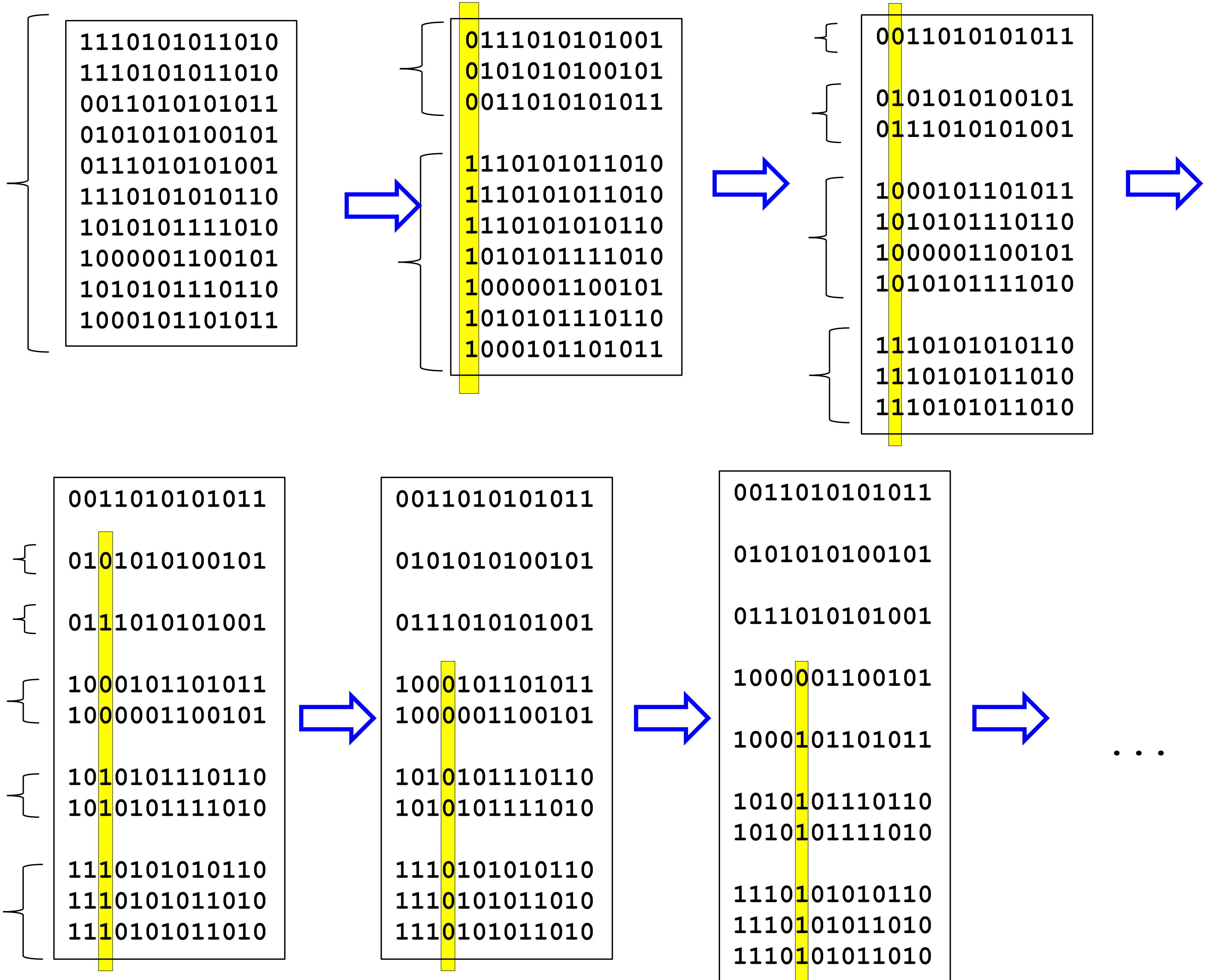


Sorting Complete Binary Strings

```
func radixESort(S array;
                k,l,r: integer) {
  if l≥r or k>m then
    return;
  end if;
  d := divide*(S, k, l, r);
  radixESort(S, k+1, l, d-1);
  radixESort(S, k+1, d, r);
}
```



- We can repeat the same procedure on the second, third, … position
- When sorting the k'th position, we only sort within the subarrays with same values in the (k-1) first positions
  - Let m by the length (in bits) of the values in S
  - Call with **`radixESort(S,1,1,|S|)`**



1110101011010
1110101011010
0011010101011
0101010100101
0111010101001
1110101010110
1010101111010
1000001100101
1010101110110
1000101101011
0111010101001
0101010100101
0011010101011
1110101011010
1110101011010
1110101010110
1010101111010
1000001100101
1010101110110
1000101101011
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0011010101011
0101010100101
0111010101001
1000001100101
1000101101011
1010101110110
1010101111010
1110101010110
1110101011010
1110101011010
. . .

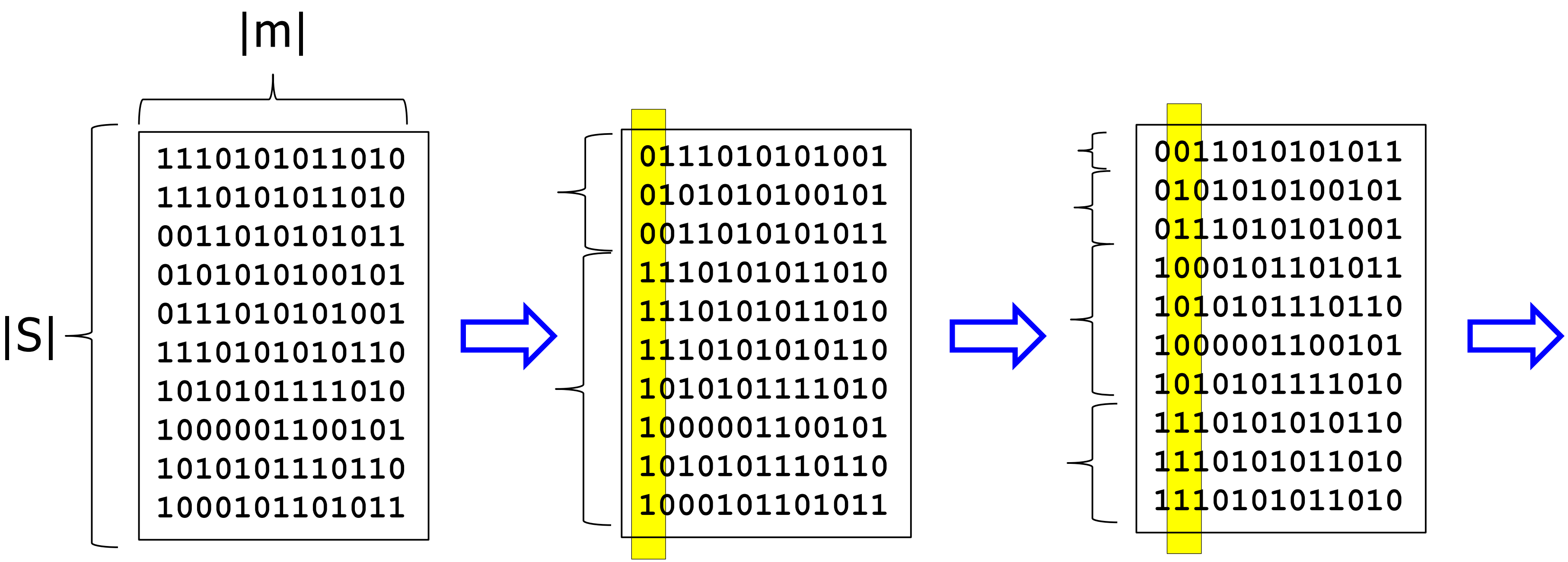
Complexity

```
func radixESort(S array;
                    k,l,r: integer) {
  if l≥r or k>m then
    return;
  end if;
  d := divide*(S, k, l, r);
  radixESort(S, k+1, l, d-1);
  radixESort(S, k+1, d, r);
}
```

```
func int divide*(S array;
                    k,l,r: int) {
  …
  repeat
    while S[i][k]=0 and i<j do
      i := i+1;
    end while;
    while S[j][k]=1 and i<j do
      j := j-1;
    end while;
    swap(S[i], S[j]);
  until i=j;
  …
  return j;   // first "1" }
```

- Total number of comparisons
  - In divide*, we look at every element S[l…r] exactly once: (r-l)
  - Then we divide S[l…r] in two disjoint halves
    - 1^{st} makes (d-l) comps
    - 2^{nd} makes (r-d) comps
  - The first call to radixESort has O(n) comps, with |S|=n.
- Are we in O(n)?

Illustration



- For every k, we look at every S[i][k] once to see whether it is 0 or 1 – together, we have at most m*|S| comparisons
  - Of course, we can stop at every interval with (r-l)=1
  - m*|S| is the worst case

Complexity (Correct)

```
func radixESort(S array;
                k,l,r: integer) {
  if l≥r or k>m then
    return;
  end if;
  d := divide*(S, k, l, r);
  radixESort(S, k+1, l, d-1);
  radixESort(S, k+1, d, r);
}
```

```
func int divide*(S array;
                  k,l,r: int) {
  …
  repeat
    while S[i][k]=0 and i<j do
      i := i+1;
    end while;
    while S[j][k]=1 and i<j do
      j := j-1;
    end while;
    swap(S[i], S[j]);
  until i=j;
  …
  return j;   // first "1" }
```

- We count …
  - Every call to radixESort first performs (r-l) comps and then divides S[l…r] in two disjoint halves
    - 1^{st} makes (d-l) comps
    - 2^{nd} makes (r-d) comps
- First call to radixESort has O(n) comps, with |S|=n
- Recursion depth is fixed to m
- Thus: O(m*|S|) comps

Some Additional Advantages

- It may not have to examine all the bits/positions

```
0011010101011
0101010100101
0111010101001
1000001100101
1000101101011
1010101110110
1010101111010
1110101010110
1110101011010
1110101011010
```

```
= 2 / 13
= 3 / 13
= 3 / 13
= 5 / 13
= 5 / 13
= 11 / 13
= 11 / 13
= 10 / 13
= 13 / 13
= 13 / 13
```

~58% of bits examined

- It works for variable length bitstrings (equal bits have to be at equal positions or padded with 0)

```
00110101
0101010100101
01110
10000011
10001011
10101011101
10101011110
11101010101
```

RadixESort or QuickSort?

- Assume we have data that can be represented as bitstrings such that more important bits are left (or right – but consistent)
  - Integers, strings, bitstrings, …
  - Equal length is not necessary, but „the same" bits must be at the same position in the bitstring (otherwise, one may pad with 0)
- Decisive: m <? >? log(n)
  - If S is large / maximal bitstring length is small: RadixESort
  - If S is small / maximal bitstring length is large: QuickSort

Content of this Lecture

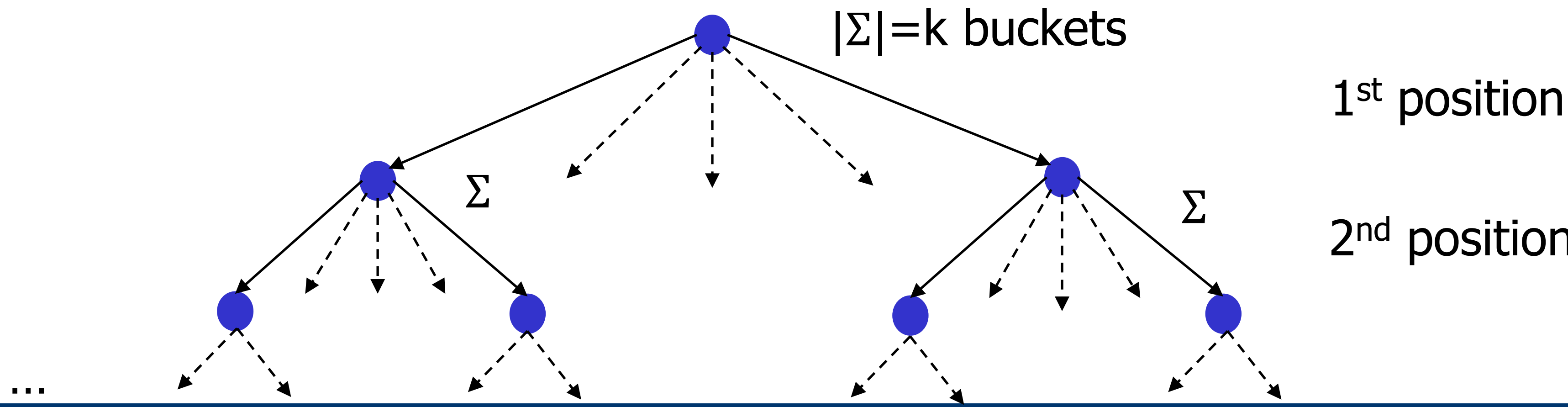
- Radix Exchange Sort
- Bucket Sort
  - Generalizing the Idea of Radix Exchange Sort to arbitrary alphabets

Bucket Sort

- What about sorting names?
- Representing "normal" Strings as bitstrings is a bad idea
  - One byte per character -> 8*length bits (large m for RadixESort)
  - But: There are only ~26 different values (no case)
- One could find shorter encodings – we go a different way

Bucket Sort generalizes RadixESort

- Assume |S|=n, m being the length of the largest value, alphabet Σ with |Σ|=k and lexicographical order (e.g., "A" < "AA")
- We first sort S on first position into k buckets (with a single scan)
- Then sort every bucket again for second position, etc.
- After at most m iterations, we are done
- Time complexity (ignoring space issues): O(m*n)
- But space is an issue



Space in Bucket Sort

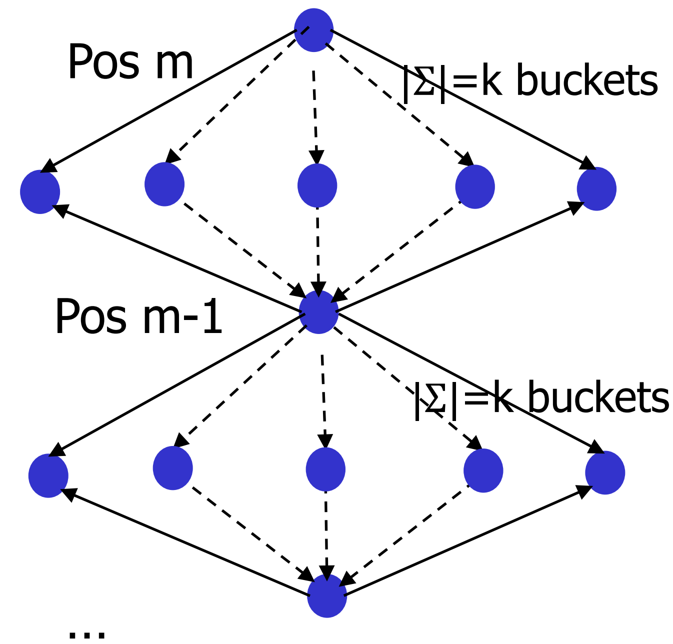
- A naïve array-based implementation allocates $k * |S|$ values for every phase of sorting into buckets
  - We do not know how many values start with a given character
  - Can be anything between 0 and |S|
- This would need to allocate a total of $O(k^m * |S|)$ space for m iterations – too much!

- We reduce this to $O(k + |S|)$
  - Requires a stable sorting algorithm for single characters
  - 1-phase of Bucket Sort is stable (if implemented our way)

Bucket Sort - Idea

- If we sort from back-to-front and keep the order of once sorted suffixes, we can (re-)use the additional space
  - Order was not preserved in RadixESort, but there we could sort in-place (only 2 values) – other problems

```
func bucketSort(S array, m,k: integer){
  B:= Array of Queues with |B|=k
  for i := m down to 1 do
      # use stable sorting algorithm
        to sort S on position i
      # merge all buckets
  end for
}
```



Caution: This is not yet $O(k+|S|)$ space …

Bucket Sort

- If we sort from back-to-front and keep the order of once sorted suffixes, we can (re-)use the additional space

```
GTT AAC GCT ATA AAC TGA TCT TTA TGG GTA TAG GGA CCG GAC GTA CAC

ATA TGA TTA GTA GGA GTA
AAC AAC GAC CAC
TGG TAG CCG
GTT GCT TCT
```

Bucket Sort

- If we sort from back-to-front and keep the order of once sorted suffixes, we can (re-)use the additional space

GTT AAC GCT ATA AAC TGA TCT TTA TGG GTA TAG GGA CCG GAC GTA CAC

ATA TGA TTA GTA GGA GTA

AAC AAC GAC CAC

TGG TAG CCG

GTT GCT TCT

ATA TGA TTA GTA GGA GTA | AAC AAC GAC CAC | TGG TAG CCG | GTT GCT TCT

Bucket Sort

- If we sort from back-to-front and keep the order of once sorted suffixes, we can (re-)use the additional space

```
GTT AAC GCT ATA AAC TGA TCT TTA TGG GTA TAG GGA CCG GAC GTA CAC

ATA TGA TTA GTA GGA GTA | AAC AAC GAC CAC | TGG TAG CCG | GTT GCT TCT

ATA TGA TTA GTA GGA GTA AAC AAC GAC CAC TGG TAG CCG GTT GCT TCT

AAC AAC GAC CAC TAG | CCG GCT TCT | TGA GGA TGG | ATA TTA GTA GTA GTT

AAC AAC GAC CAC TAG CCG GCT TCT TGA GGA TGG ATA TTA GTA GTA GTT

AAC AAC ATA | CAC CCG | GAC GCT GGA GTA GTA GTT | TAG TCT TGA TGG TTA

AAC AAC ATA CAC CCG GAC GCT GGA GTA GTA GTT TAG TCT TGA TGG TTA
```

Bucket Sort – Pseudocode (aka LSD Radix Sort)

- Sort S from back-to-front
  - (Re-)use k queues, one for each bucket
    - findBucket translates the i-th char of S[j] into a bucket
  - E.g. map ‚A-Z‘ to 1-26
  - The number of queues must be equal to k
    - Avoid large alphabets …
- Stable: Append to end of queue
- Finally, merge buckets and continue with next position

```
func bucketSort(S array,
                         m, k: integer){
  B:= Array of Queues with |B|=k
  for i := m down to 1 do
    for j := 1 to |S| do
     k := findBucket(S[j][i]);
     B[k].enqueue(S[j]);
    end for
    j := 1;
    for k := 1 to |B| do
      while not B[k].isEmpty() do
        S[j] := B[k].dequeue();
        j := j + 1;
      end while end for
  end for
  return S;
}
```

Magic? Proof

- By induction
- Assume that before phase t we have sorted all values by the (t-1)-suffix (right-most, least important for order)
  - True for t=2 – we sorted by the last character ((t-1)-suffixes)
- In phase t, we sort by the t'th lowest value (from the right)
- This will group all values from S with the same value in S[i][m-t+1] together and keep them sorted wrt. (t-1)-suffixes
  - Assuming a stable sorting algorithm
- Since we sort by S[i][m-t+1], the array after phase t will be sorted by the t-suffix
- qed.

Saving More Space

- The example has shown that we actually never need more than $O(|S|+k)$ additional space (all buckets together)
  - Use a linked-list/queue for each bucket
  - Keep pointer to start (for copying) and end (for extending) of each list – this requires 2*k space
  - All lists together only store |S| elements (of length m)

A Word on Names

- Names of these algorithms are not consistent
  - Radix Sort generally depicts the class of sorting algorithms which look at single keys and partition keys in smaller parts
  - RadixESort is also called binary quicksort (Sedgewick)
  - Bucket Sort is also called „Sortieren durch Fachverteilen" (OW), RadixSort (German WikiPedia and Cormen et al.), or LSD Radix Sort (Sedgewick), or distribution sort
  - Cormen et al. use Bucket Sort for a variation of our Bucket Sort (linear only if keys are equally distributed)
  - …

Summary

	Comps worst case	avg. case	best case	Additional space	Moves (wc / ac)
Selection Sort	$O(n^2)$		$O(n^2)$	O(1)	O(n)
Insertion Sort	$O(n^2)$		O(n)	O(1)	$O(n^2)$
Bubble Sort	$O(n^2)$		O(n)	O(1)	$O(n^2)$
Merge Sort	O(n*log(n))		O(n*log(n))	O(n)	O(n*log(n))
QuickSort	$O(n^2)$	O(n*log(n)	O(n*log(n)	O(log(n))	$O(n^2)$ / O(n*log(n))
BucketSort (m= …)	O(m*(n+k))			O(n+k)	

Exemplary Questions

- What is the best case complexity of BucketSort?
- What is the space complexity of RadixESort?
- What is a stable sorting algorithm?
- Which of the following sorting algorithms are stable: BubbleSort, InsertionSort, MergeSort?
- BucketSort needs a data structure for building and using buckets. Give an implementation using (a) a heap, (b) a queue.