Algorithms and Data Structures

One Problem, Four Algorithms

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Content of this Lecture

- The Max-Subarray Problem
- Naïve Solution
- Better Solution
- Best Solution

Where is the Sun?

Source: http://www.layoutsparks.com

How can we find the Sun Algorithmically?

- Assume pixel (RGB) representation
- The sun obviously is bright
- RGB colors can be transformed into brightness scores
- The sun is the brightest spot
  - Compute an average brightness for the entire picture
  - Subtract this from each brightness value (will yield negative values)
  - Find the shape (spot) such that the sum of its brightness values is maximal

Size of the Spot not Pre-Determined

Example (Shapes: only Rectangles)

1	6	8	6	5	3
7	9	5	4	2	2
2	7	6	3	2	1
1	3	0	0	0	1
2	4	8	8	3	2
3	7	9	8	8	3

Avg. ~4

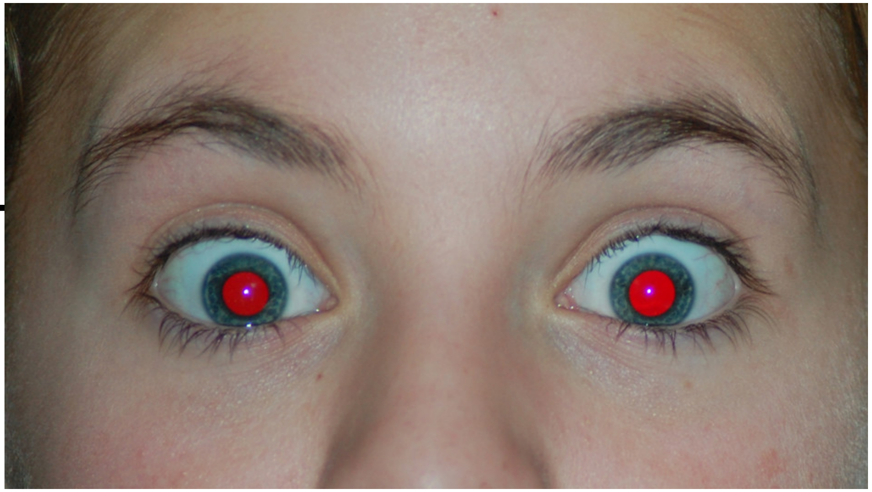
-3	2	4	2	1	-1
3	5	1	0	-2	-2
-2	3	2	-1	-2	-3
-3	-1	-4	-4	-4	-3
-2	0	4	4	-1	-2
-1	3	5	4	4	-1

-3	**2**	**4**	2	1	-1
3	**5**	**1**	0	-2	-2
-2	**3**	**2**	-1	-2	-3
-3	-1	-4	-4	-4	-3
-2	0	4	4	-1	-2
-1	3	5	4	4	-1

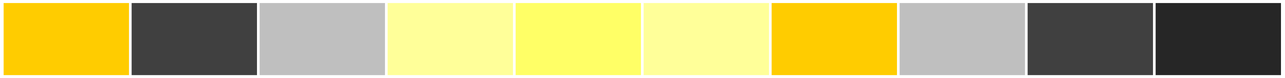
-3	**2**	**4**	**2**	1	-1
3	**5**	**1**	**0**	-2	-2
-2	**3**	**2**	**-1**	-2	-3
-3	-4	-4	-4	-3	-3
-2	0	4	4	-1	-2
-1	3	5	4	4	-1

-3	2	4	2	1	-1
3	5	1	0	-2	-2
-2	3	2	-1	-2	-3
-3	-4	-4	-4	-3	-3
-2	**0**	**4**	**4**	**-1**	-2
-1	**3**	**5**	**4**	**4**	-1

Simpler Problem



- This is a bit complicated
  - Which shapes?
  - Shape should not be too big (sun is small compared to sky)
    - What if the sun is almost filling the picture?
  - Maximal sum of scores or maximal average score?
  - (see very last slide)
- We look at a simpler problem: Max Subarray
  - Where is the sun?



Max-Subarray Problem

- Definition (Max-Subarray Problem)
  Assume an array A of integers. Find the highest sum-score s of all subarrays A* of A, where the sum-score of an array A* is the sum of all its values. If s* is negative, return 0*
- Remarks
  - Cells may have positive or negative values (or 0)
  - We only want the maximal value, not the borders of A*
  - There might be multiple A*, but only one max sum-score
  - Length of the subarray A* is not fixed (shape of spot)

-2	0	4	3	4	-6	-1	12	-2	0	15

A Greedy Solution

- Promising start point: Find maximal value in array A
- Greedy: Expand in both directions until sum decreases
- Complexity?

A Greedy Solution

- Promising start point: Find maximal value in array A
- Greedy: Expand in both directions until sum decreases
- Complexity? (Let n=|A|)
  - O(n) to find maximal value
  - O(n) expansion steps in worst case
  - O(n) together
- Do we optimally solve our problem?

A Greedy Solution

- Promising start point: Find maximal value in array A
- Greedy: Expand in both directions until sum decreases
- Complexity? (Let n=|A|)
  - O(n) together
- Do we optimally solve our problem?

-2	0	4	3	4	-3	-1	12	2	-1	1

-2	0	4	3	4	-3	-1	12	2	-1	1

-2	0	4	3	4	-3	-1	12	2	-1	1

A Greedy Solution

- Promising start point: Find maximal value in array A
- Greedy: Expand in both directions until sum decreases
- Complexity? (Let n=|A|)
  - O(n) together
- Do we optimally solve our problem?

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-2	0	4	3	4	-3	-1	12	2	-1	1

-2	0	4	3	4	-3	-1	12	2	-1	1

- First step may already be wrong

-2	0	4	3	4	-6	-6	10	-6	-1	1

Content of this Lecture

- The Max-Subarray Problem
- Naïve Solution
- Better Solution
- Best Solution

Naive Solution: Look at all Subarrays

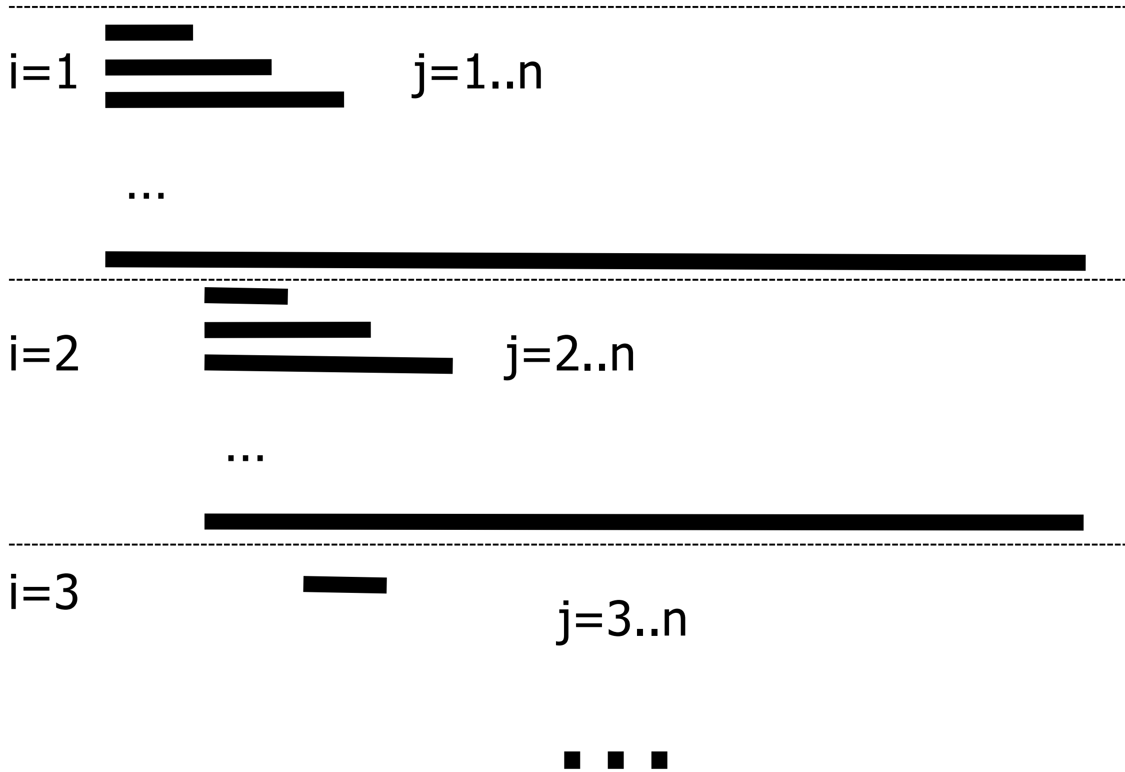
```
A: array_of_integer;
n := |A|;
m := 0;
for i := 1 … n do
  for j := i … n do
    s := 0;
    for k := i … j do
      s := s + A[k];
    end for;
    if s>m then
      m := s;
    end if;
  end for;
end for;
return m;
```

- i: Every start point of an array
- j: Every end point of an array
- k: Compute the sum of the values between start and end

Illustration

```
A: array_of_integer;
n := |A|;
m := 0;
for i := 1 … n do
  for j := i … n do
    s := 0;
    for k := i … j do
      s := s + A[k];
    end for;
    if s>m then
      m := s;
    end if;
  end for;
end for;
return m;
```

-2	0	4	3	4	-3	-1	12	2	-1	1



Complexity

```
A: array_of_integer;
n := |A|;
m := 0;
for i := 1 … n do
  for j := i … n do
    s := 0;
    for k := i … j do
      s := s + A[k];
    end for;
    if s>m then
      m := s;
    end if;
  end for;
end for;
return m;
```

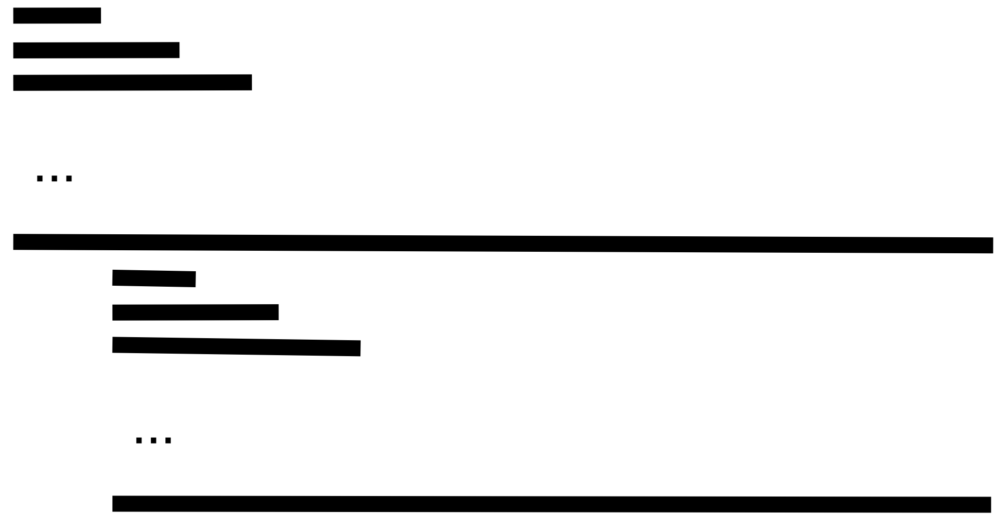
- Complexity?
- i-loop: n times
- j-loop: n times (worst-case)
  - Together ~$n^2/2$, which is in $O(n^2)$
- Inner loop: n times
- Together: $O(n^3)$
- But: We are summing up the same numbers again and again
- We perform redundant work
- More clever ways?

Exhaustive Solution

- First sum: A[1]
- Second: A[1]+A[2]
- 3rd: A[1]+A[2]+A[3]
- 4th: …

- Every next sum (k) is the previous sum plus the next cell (j)
- How can we reuse the previous sum?

-2	0	4	3	4	-3	-1	12	2	-1	1



Exhaustive Solution, Improved

- Every next sum is the previous sum plus the next cell
- Complexity: $O(n^2)$

```
A: array_of_integer;
n := |A|;
m := 0;
for i := 1 … n do
  s := 0;
  for j := i … n do
    s := s + A[j];
    if s>m then
      m := s;
    end if;
  end for;
end for;
return m;
```

Content of this Lecture

- The Max-Subarray Problem
- Naïve Solution
- Better Solution
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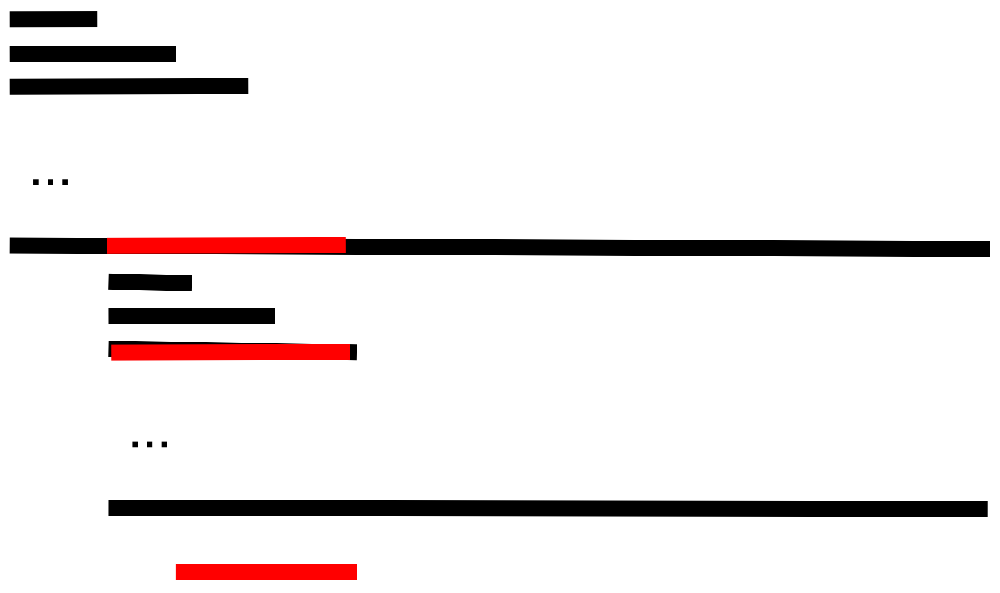
Observation

- We optimized computation of sums in the j/k looks
- We still compute many sums multiple times – across i's

-2	0	4	3	4	-3	-1	12	2	-1	1



Divide and Conquer

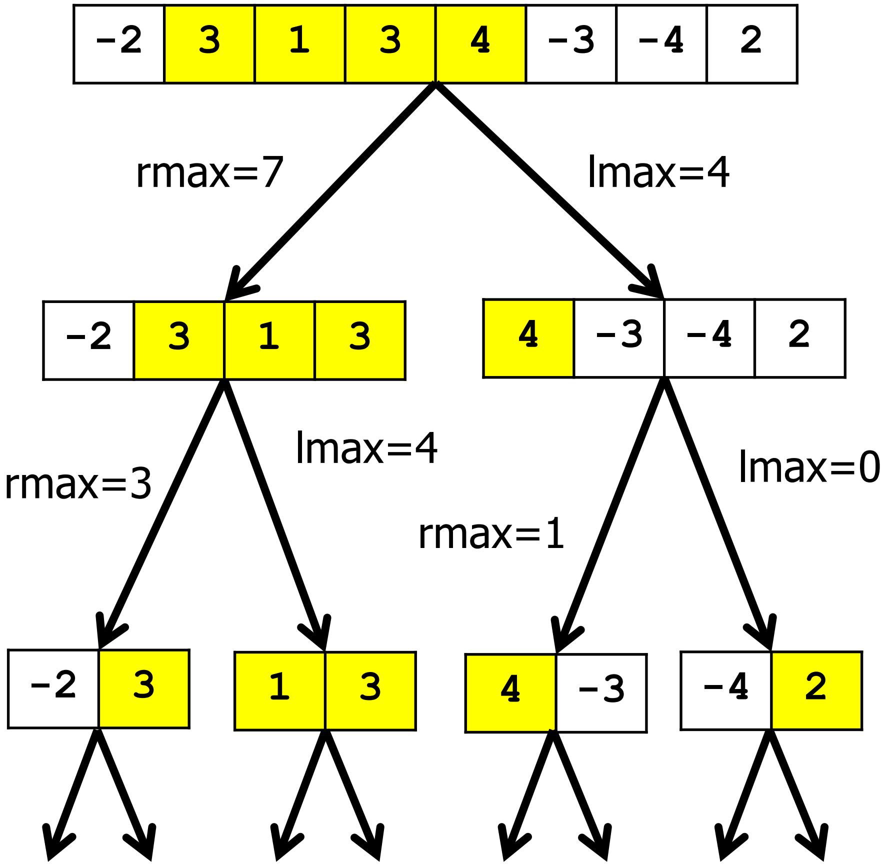
- We can break up our problem into smaller ones by looking only at parts of the array
- One scheme: Assume $A=A_1|A_2$
  - With "|" meaning array concatenation and $|A_1|=|A_2|(+0/1)=|A|/2$
- The max-subarray (msa) of A …
  - either lies in A_1 – can be found by solving msa(A_1)
  - or in A_2 – can be found by solving msa(A_2)
  - or partly in A_1 and partly in A_2
    - Can be solved by summing-up the msa's in A_1/A_2 that align with the right/left end of A_1/A_2
- We divide the problem into smaller ones and create the "bigger" solution from the "smaller" solutions

Algorithm (for simplicity, assume $|A|=2^x$ for some x)

```
function msa (A: array_of_int) {
  n := |A|;
  if (n=1) then
    if A[1]>0 then
      return A[1]
    else
      return 0;
  end if;
  m := n/2;
  A1 := A[1…m];
  A2 := A[m+1…n];
  l1 := rmax(A1);
  l2 := lmax(A2);
  m := max(msa(A1),
           l1+l2,
           msa(A2));
  return m;
}
```

```
function rmax (A: array_of_int){
  n := |A|;
  s := 0;
  m := 0;
  for i := n .. 1 do
    s := s + A[i];
    if s>m then
      m := s;
    end if;
  end for;
  return m;
}
```

Example



- Solution: max(7,7+4,4)
- Left array: max(3,3+4,4)
- Right array: max(4,1+0,2)
- Left-most: max(0,0+3,3)
- …

Complexity

- This time it is not so easy …
- Complexity of lmax / rmax?

```
function rmax (A: array_of_int){
  n := |A|;
  s := 0;
  m := 0;
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  end for;
  return m;
}
```

Complexity

- This time it is not so easy …
- Complexity of lmax / rmax?
  - O(n)
- Function msa
  - Let T(n) be the number of steps necessary to execute the algorithm for |A|=n
    - In each level, n'=n/2
    - The two sub-solutions require T(n') each
  - This yields: T(n) ~ O(1)+O(n)+T(n/2)+T(n/2)

```
function msa (A: array_of_int) {
  n := |A|;
  if (n=1) then
    if A[1]>0 then
      return A[1]
    else
      return 0;
  end if;
  m := n/2;    # …
  A1 := A[1…m];
  A2 := A[m+1…n];
  l1 := rmax(A1);
  l2 := lmax(A2);
  m := max(msa(A1),l1+l2,msa(A2));
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Complexity

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  - O(n)
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Complexity

```
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  n := |A|;
  if (n=1) then
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      return A[1]
    else
      return 0;
  end if;
  m := n/2;    # Assume even sizes
  A1 := A[1…m];
  A2 := A[m+1…n];
  l1 := rmax(A1);
  l2 := lmax(A2);
  m := max( msa(A1), l1+l2, msa(A2));
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```

- For constants c_1, c_2
- $T(n) = 2*T(n/2) + c_1*n$
- Further: $T(1) = c_2$

Complexity

```
function msa (A: array_of_integer) {
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    if A[1]>0 then
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      return 0;
  end if;
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  A1 := A[1…m];
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  l1 := rmax(A1);
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```

- For constants c_1, c_2
- $T(n) = 2*T(n/2)+c_1*n$
- Further: $T(1) = c_2$
- Iterative substitution:

$$T(n)= 2*T(n/2)+c_1n =$$

$$= 2(2T(n/4)+c_1n/2)+c_1n \quad = 4T(n/4)+2c_1n=$$

$$= 4(2T(n/8)+c_1n/4)+2c_1n = \underbrace{8T(n/8)}_{2^{\log(n)}*c_2}+\underbrace{3c_1n}_{c_1n*\log(n)} = \ldots$$

$$2^{\log(n)}*c_2 + c_1n*\log(n) =$$

$$c_2n+c_1n*\log(n) \quad = O(n*\log(n))$$

Same Problem, Different Algorithms

- Naive: $O(n^3)$
- Less naive, still redundant: $O(n^2)$
- Divide & Conquer: $O(n*\log(n))$

- The problem: $O(n)$

Content of this Lecture

- The Max-Subarray Problem
- Naïve Solution
- Better Solution
- Linear Solution

Let‘s Think again – More Carefully

- Let‘s use another strategy for dividing the problem
- Let‘s look at the solutions for A[1], A[1..2], A[1…3], …
- What can we say about the msa for A^{i+1}=A[1…i+1], given the msa of A^i=A[1…i]?

-2	0	4	3	4	-3	-1	6

Let‘s Think again – More Carefully

- Let‘s use another strategy for dividing the problem
- Let‘s look at the solutions for A[1], A[1..2], A[1…3], …
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-2	0	4	3	4	-3	-1	6

- msa(A^{i+1}) is …
  - either somewhere within A^i, which means the same as msa(A^i)
  - or formed by rmax(A^i)+A[i+1]
- Idea: Keep msa and rmax while scanning once through A

Algorithm & Complexity

```
A: array_of_integer;
rmax:= 0;
m := 0;
for i:= 1 to n do
  rmax := max( 0, A[i],
               rmax+A[i]);
  m := max( rmax, m);
end for;
```

- Obviously: O(n)
- Asymptotically optimal
  - We only look a constant number of times at every element of A
  - But we need to look at least once at every element of A
  - Thus, the problem is $\Omega(n)$
- Example of dynamic programming: Build larger solutions from smaller ones

Example

									rmax	m
-2	3	1	3	4	-3	-4	2		0	0
-2	3	1	3	4	-3	-4	2		3	3
-2	3	1	3	4	-3	-4	2		4	4
-2	3	1	3	4	-3	-4	2		7	7
-2	3	1	3	4	-3	-4	2		11	11
-2	3	1	3	4	-3	-4	2		8	11
-2	3	1	3	4	-3	-4	2		4	11
-2	3	1	3	4	-3	-4	2		6	11

Optimization Problems

- Optimization – find the best among all possible solutions
- Issues
  - Find solutions: Simple here, but sometimes hard
  - Score solutions: Simple here, but sometimes hard
  - Search space pruning: Do we need to look at all possible solutions?
- Typical pattern
  - Enumerate solutions in a systematic manner
  - Often generates a tree of partial and finally complete solutions
  - Prune parts of the search space where no optimal solution can be
  - If possible, stop early

Types of Algorithms

- Different fundamental patterns (non exhaustive list)
  - Greedy: Find some promising start point and expand aggressively until a complete solution is found
    - Usually fast, but usually doesn’t find the optimal solution
  - Exhaustive: Test all possible solutions and find the one that is best
    - Sometimes the only choice if optimality is asked for
  - Divide & Conquer: Break your problem into smaller ones until these are so easy that they can be solved directly; construct solutions for “bigger” problems from these small solutions
  - Dynamic programming
  - Backtracking
  - …

Types of Algorithms

- For the max subarray problem
  - Greedy: $O(n)$, but wrong
  - Exhaustive: $O(n^3)$
    - With pruning $O(n^2)$
  - Divide & Conquer: $O(n*log(n))$
  - Dynamic programming: $O(n)$
  - Backtracking
  - …
- Notes
  - No sharp way to differentiate algorithmic patterns
  - Usually there are different greedy, exhaustive, … solutions

Exemplary Questions

- Give an optimal algorithm for the max-subarray problem and prove its optimality
- Assume the max-subarray problem with the additional restriction that the length of sub-array must be short-or-equal a constant k. Give a linear algorithm solving this problem.
- Give an algorithm for the max-subarray problem in 2D, where |A| is quadratic and the subarray must be a square. Analyze its worst-case complexity.
  - Hint: For improvements, store intermediate results