Algorithms and Data Structures

Priority Queues

Ulf Leser

Specialized Queues: Priority Queues

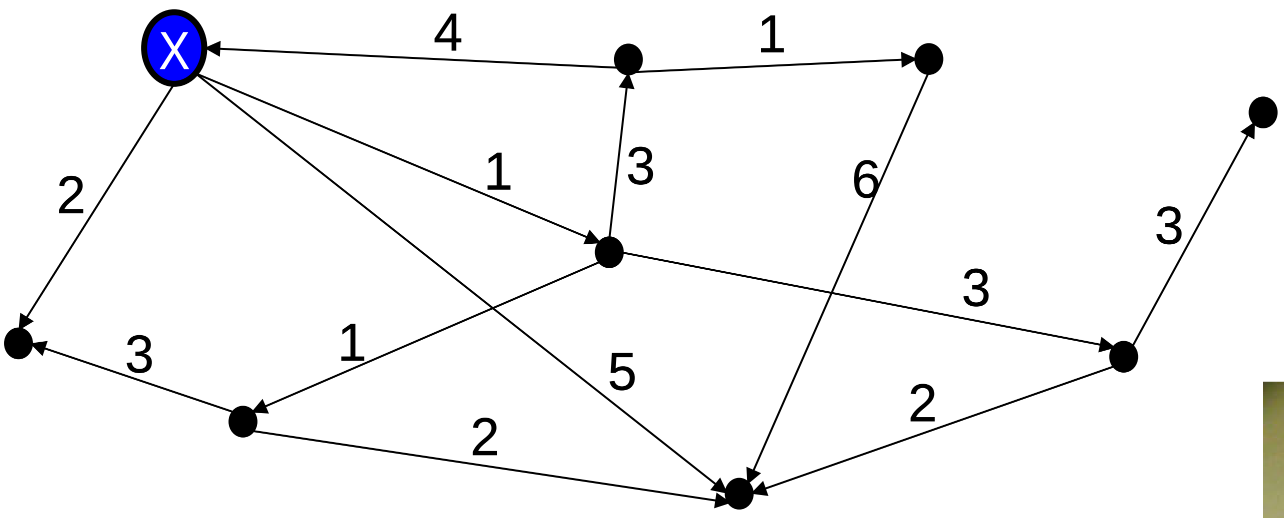
- Up to now, we assumed that all elements are equally important and that any of them could be searched next
- What if some elements are more important than others?
- In many applications, elements have a priority
  - Requests to data on disks in multi-core hardware
  - Request of memory blocks in multi-core hardware
  - Bandwidth in LANs (VoIP, streaming, ...)
  - Next best move in board games
  - ...
  - Next access always retrieves the currently most important element
- Such data structures are called Priority Queues

Differences

- Counter examples
  - Stock exchange orders
  - Bandwidth on the internet (?)
  - Very delicate topic: Fairness versus priority
- Difference to Self-Organizing Lists
  - Most important element is always retrieved next – should be O(1)
  - List should be kept ordered by priority

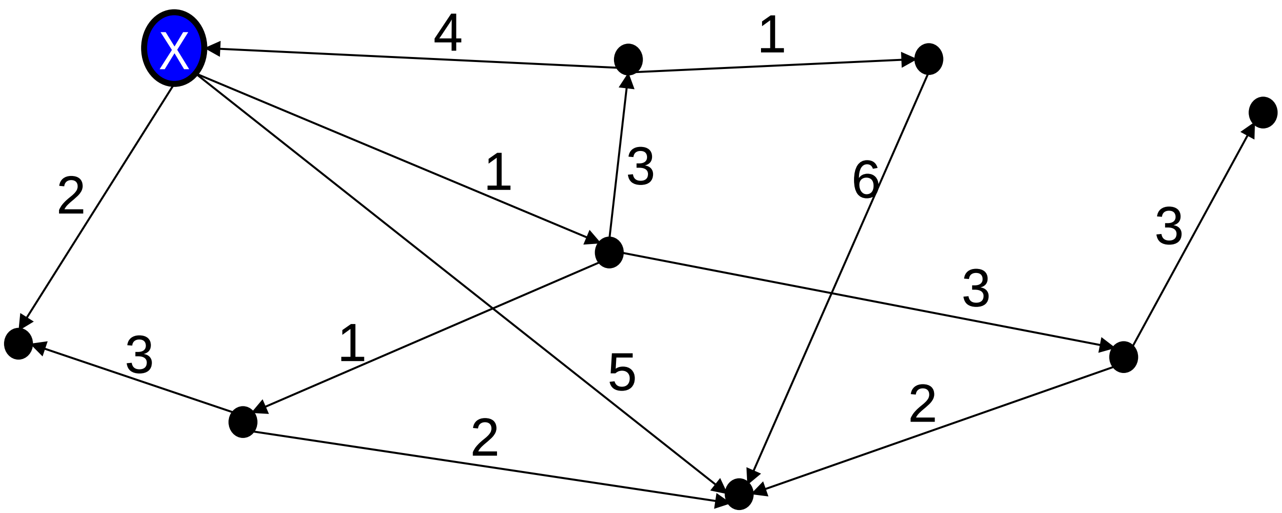
- We next look at a scenario where new elements are inserted all the time, elements may change their priority, and the most important element is removed regularly

Shortest Paths in a Graph



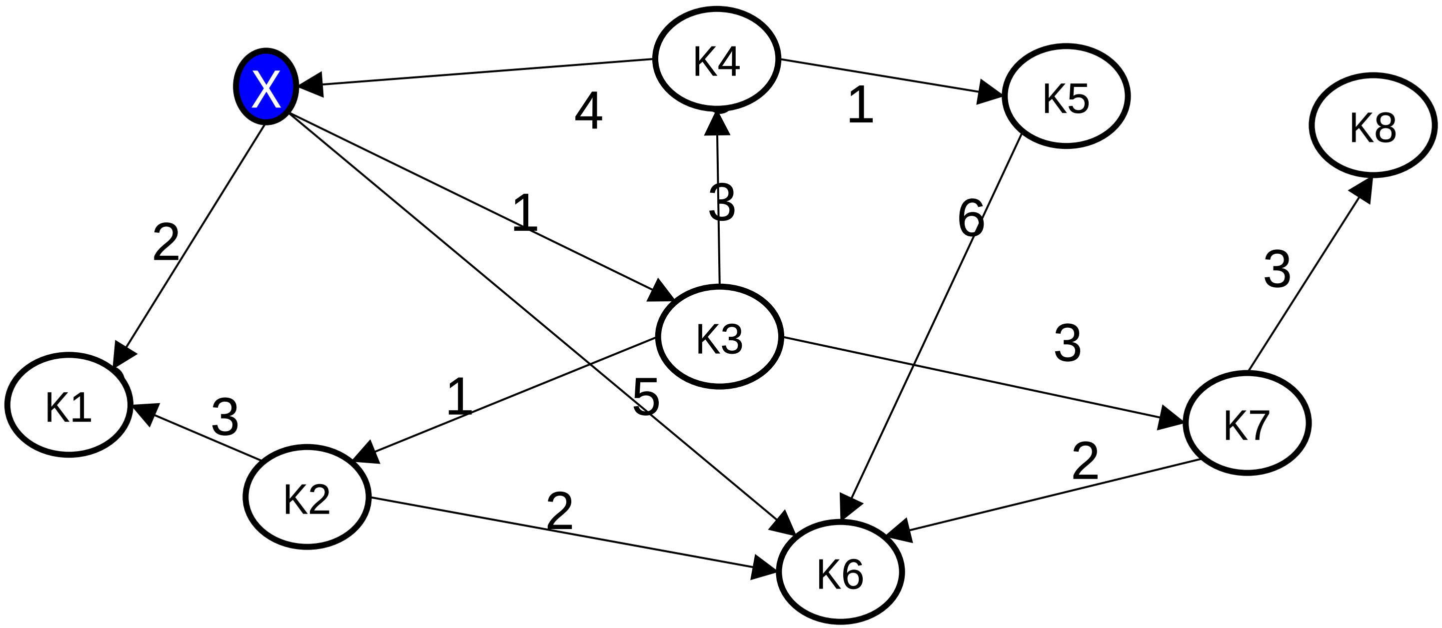
- Task: Find the distance between X and all other nodes
  - Classical problem: Single-Source-Shortest-Paths
  - Famous solution: Dijkstra’s algorithm
    - E. Dijsktra: A Note on Two Problems in Connexion with Graphs. Numerische Mathematik 1 (1959), S. 269–271

Assumptions



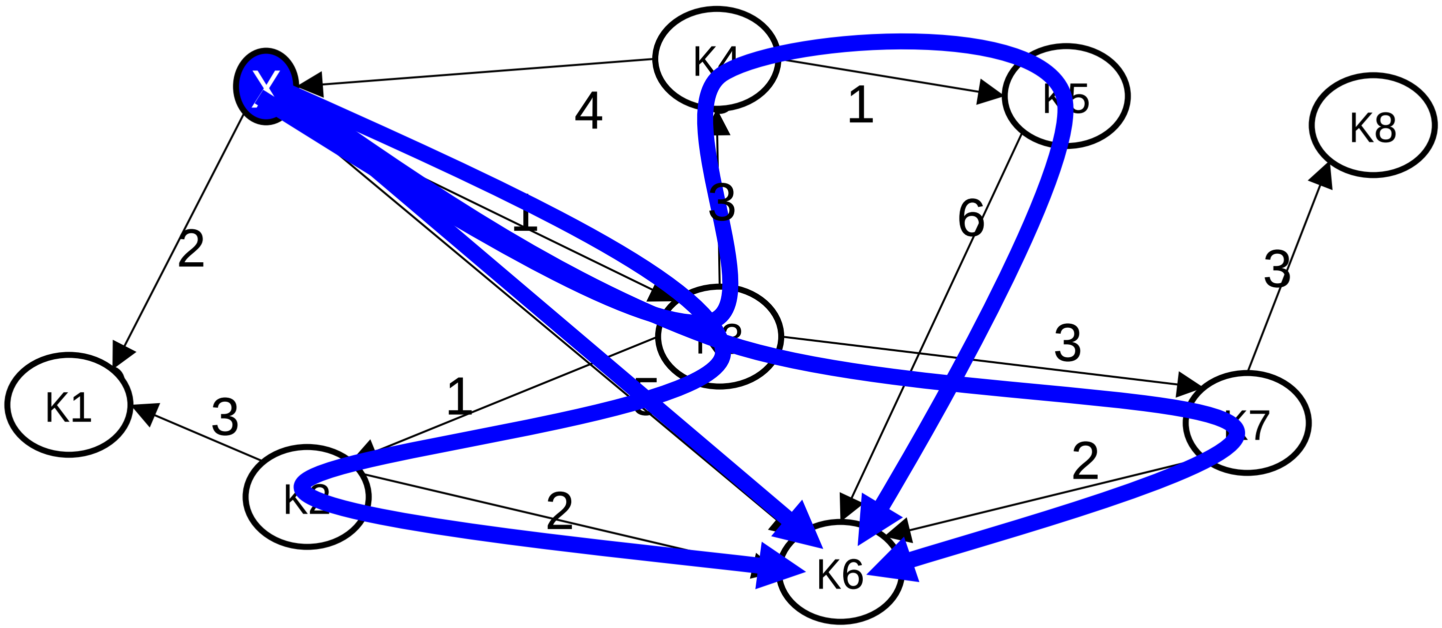
- We assume that every node is reachable from X
- Distance is the length (=sum of edge weights) of the shortest path
  - There might be many shortest paths, but distance is unique
  - We only want the distances and need no "witness paths"
- We assume strictly positive edge weights
  - Whenever we extend a path with an edge, its length increases
  - Thus, no shortest path may contain a cycle

Exhaustive Solution



- First approach: Enumerate all paths
  - Need to break cycles (e.g. X – K3 – K4 – X – K3 - ...)
  - Using DFS: X – K3 – K4 – X [BT-K4] – K5 – K6 [BT-K5] [BT-K4] [BT-K3] – K7 – K8 [BT-K7] – K6 [BT-K7] [BT-K3] – K2 – K6 [BT-K2] – K1 [BT-K2] [BT-K3] [BT-X] K6 - ...

Redundant work



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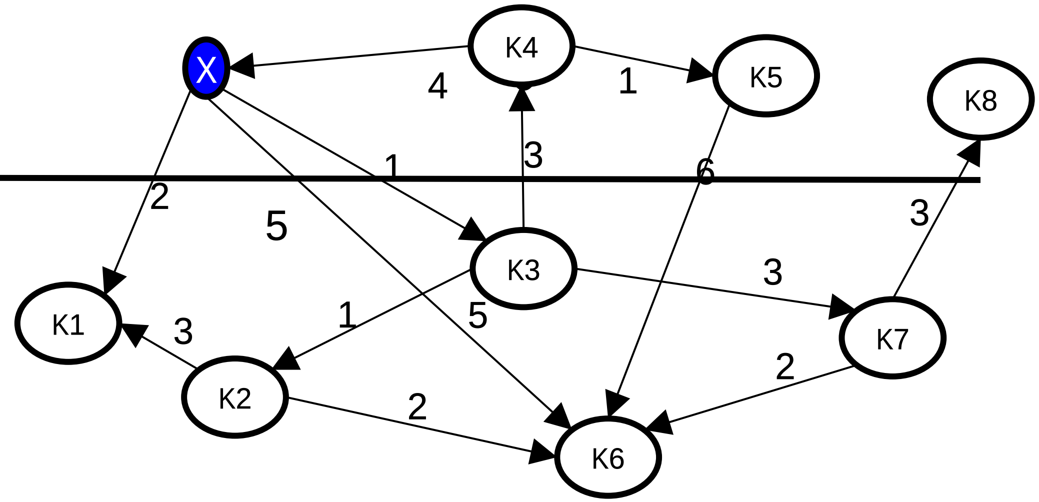
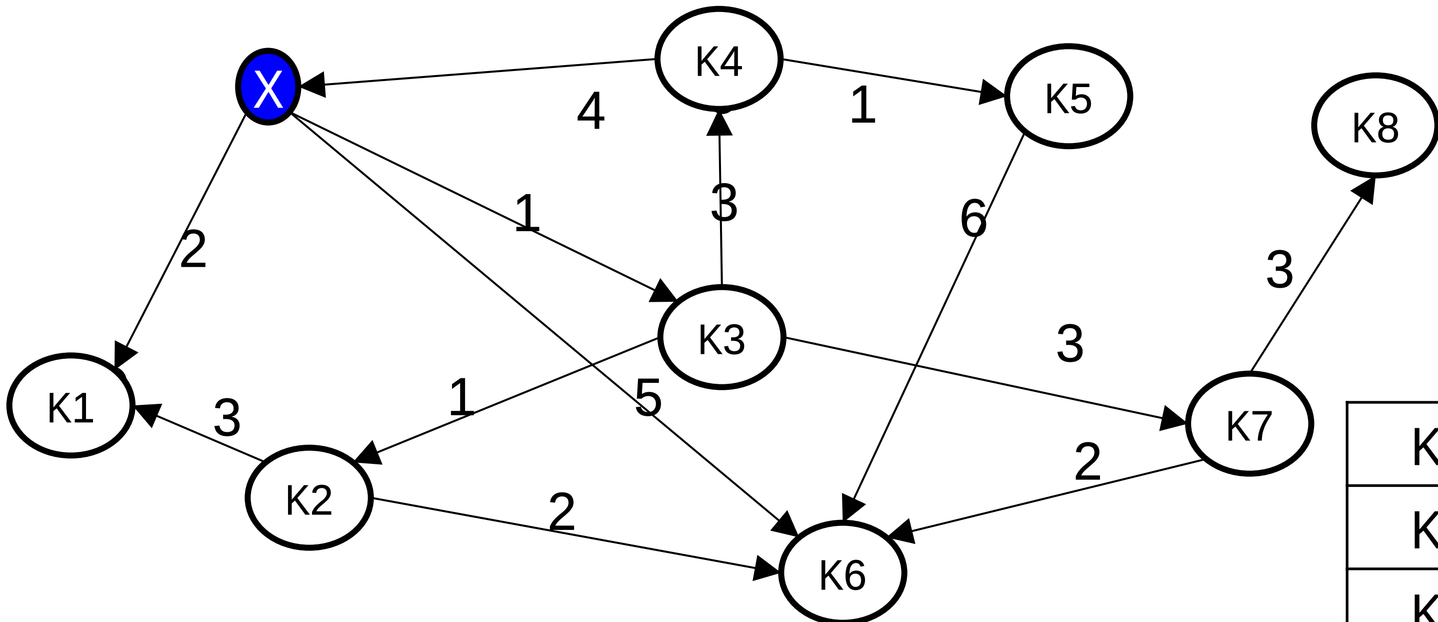
Dijkstra's Idea



- Enumerate paths from X by their length
  - Neither DFS nor BFS
- Assume we reach a node Y by a path p of length l and we have already explored all paths from X with length l' ≤ l and that Y was not reached yet
- Then p must be a shortest path between X and Y
  - Because any p' between X and Y would have a prefix of length at least l and (a) a continuation with length>0 or (b) would not need a continuation (then p is as short as p')

Example for Idea



K3	1
K2	2
K1	2
K6	4
K4	4
K7	4
K5	5
K8	7

- 1: X – K3
- 2: X – K3 – K2
- 2: X – K1
- 4: X – K3 – K2 – K6
- 4: X – K3 – K4
- 4: X – K3 – K7
- 5: X – K3 – K4 – K5
- 7: X – K3 – K7 – K8
- Stop (all nodes found)

Algorithmic Idea

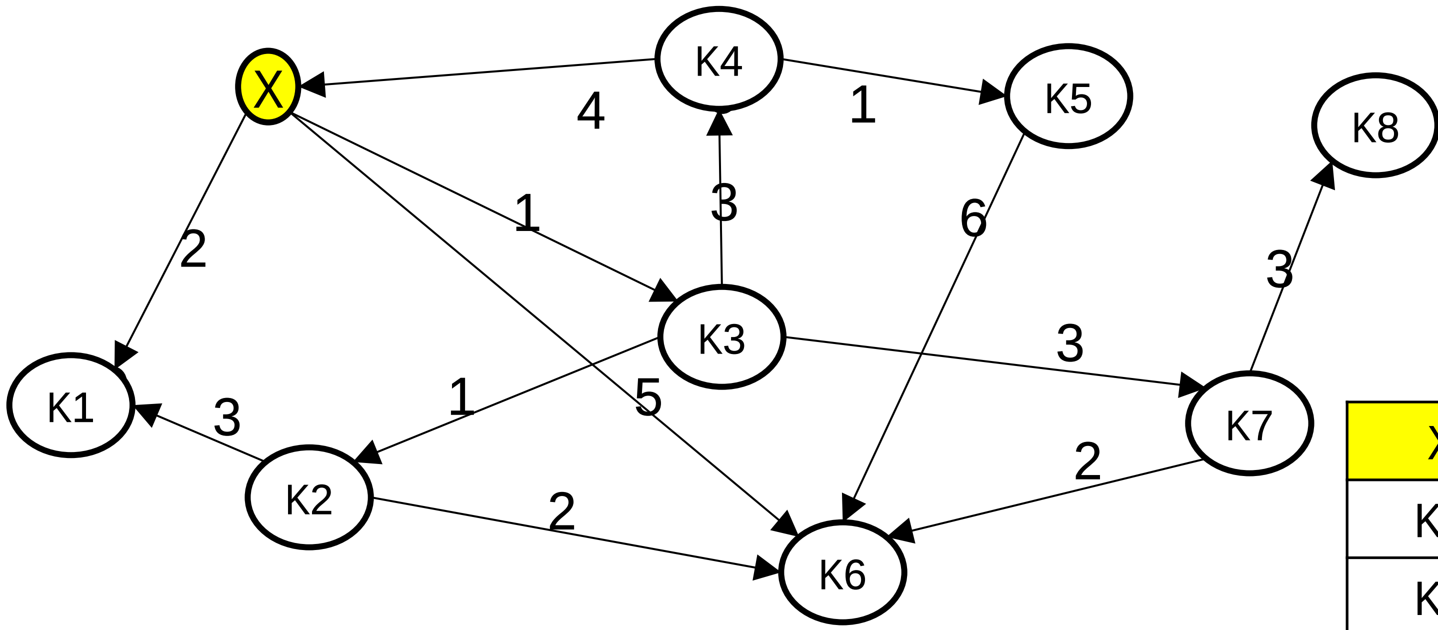
- Enumerate paths by iteratively extending already found short paths by all possible extensions
  – All edges outgoing from the end node of a short path
- These extensions
  – ... either lead to a node which we didn't reach before – then we found a path, but cannot yet be sure it is the shortest
  – ... or lead to a node which we already reached but we are not yet sure of we found the shortest path to it – update current best distance
  – ... or lead to a node which we already reached and for which we also surely found a shortest path already – these can be ignored
- Eventually, we enumerate nodes by their distance

Algorithm

```
G = (V, E);
x : start_node;    # x∈V
A : array_of_distances;
∀i: A[i]:= ∞;
L := V;
A[x] := 0;
while L≠∅
  k := L.get_closest_node();
  L := L \ k;
  forall (k,f,w)∈E do
    if f∈L then
      new_dist := A[k]+w;
      if new_dist < A[f] then
        A[f] := new_dist;
      end if;
    end if;
  end for;
end while;
```

- Assumptions
  - Nodes have IDs between 1 ... |V|
  - Edges are **`(from, to, weight)`**
- We enumerate nodes by length of their shortest paths
  - In the first loop, we pick x and update distances (A) to all adjacent nodes
  - When we pick a node k, we already have computed its distance to x in A
  - We adapt the current best distances to all neighbors of k we haven't picked yet
- Once we picked all nodes, we are done

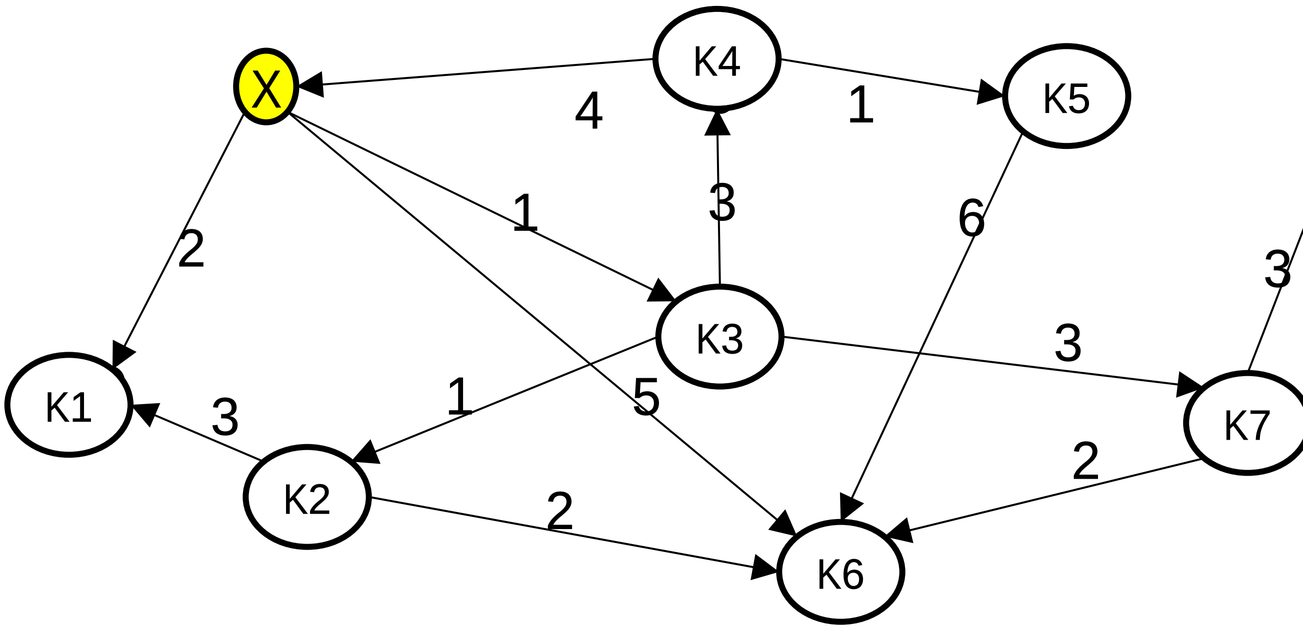
Example for Algorithm



X	0
K1	∞
K2	∞
K3	∞
K4	∞
K5	∞
K6	∞
K7	∞
K8	∞

- Pick x

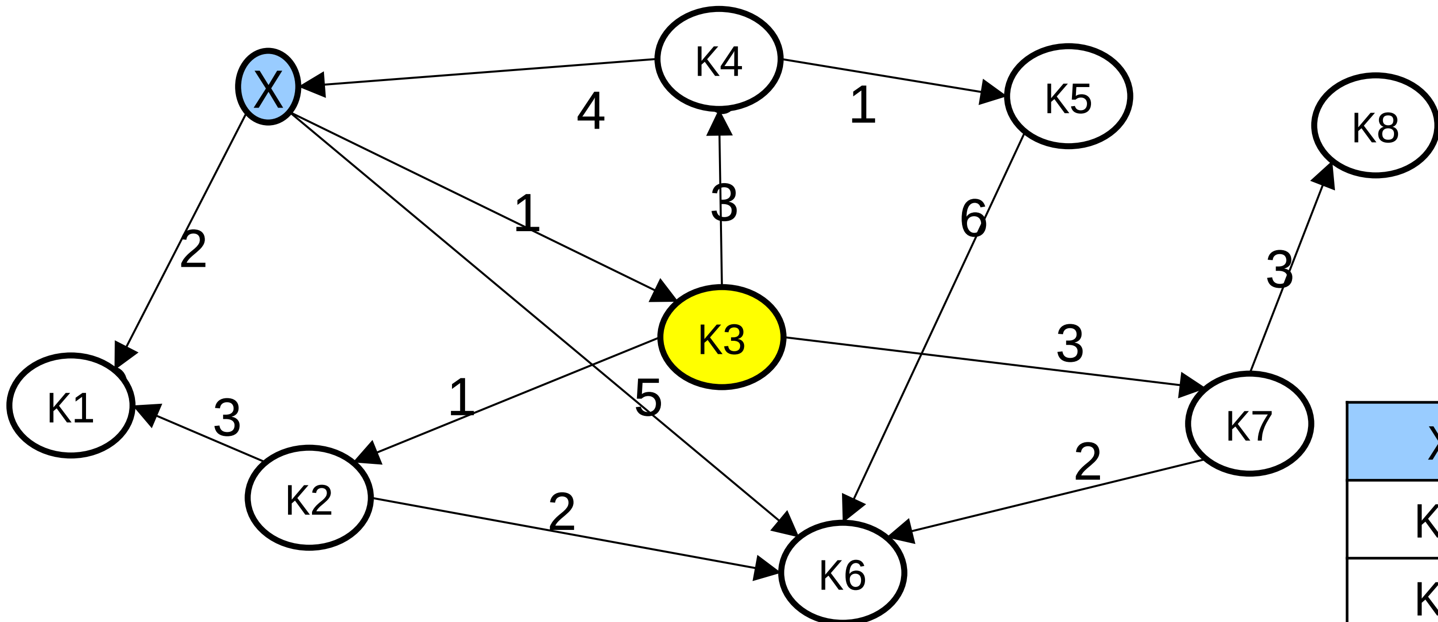
Example for Algorithm



X	0
K1	2
K2	∞
K3	1
K4	∞
K5	∞
K6	5
K7	∞
K8	∞

- Pick x
- Adapt distances to all neighbors

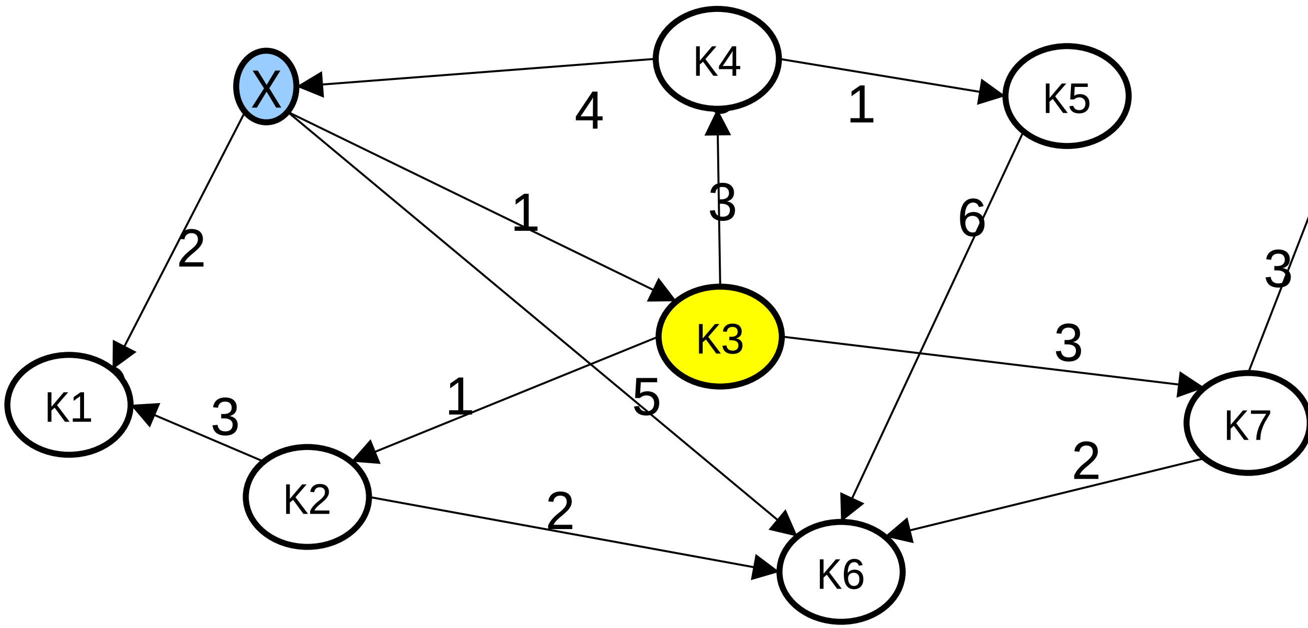
Example for Algorithm



X	0
K1	2
K2	∞
K3	1
K4	∞
K5	∞
K6	5
K7	∞
K8	∞

- Pick K3 (closest to x)

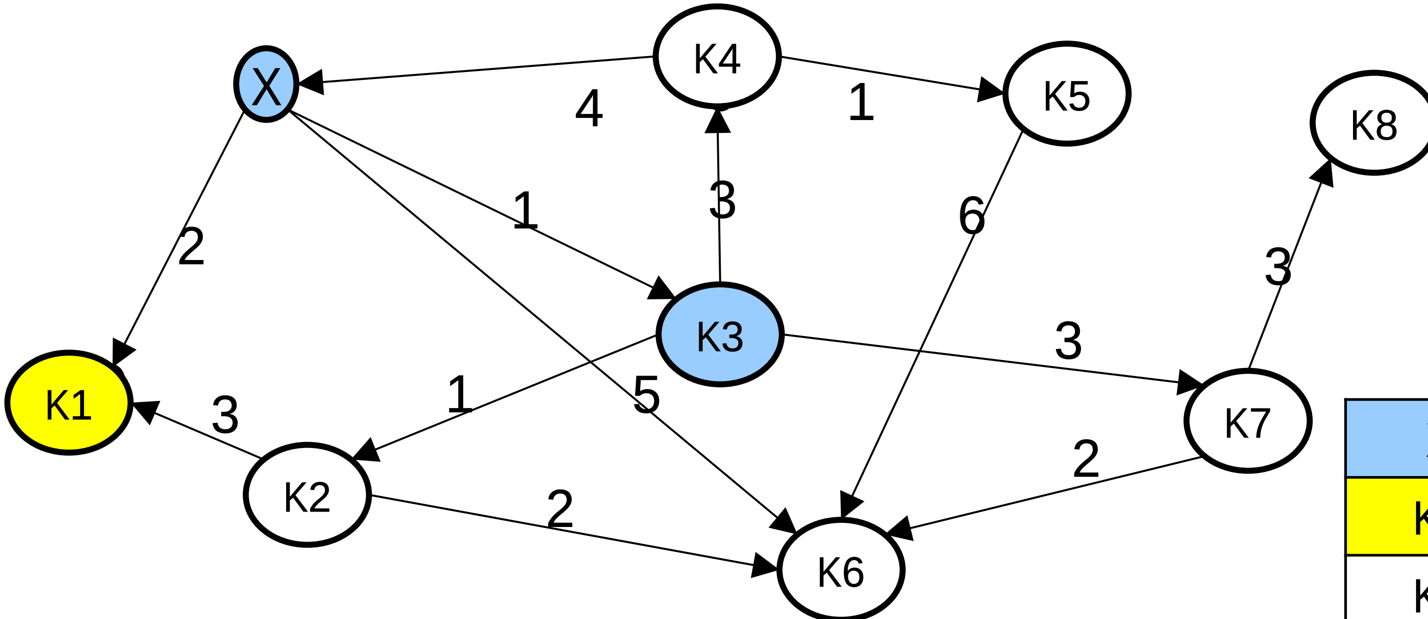
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	5
K7	4
K8	∞

- Pick K3
- Adapt distances (from x) to all neighbors (of K3)

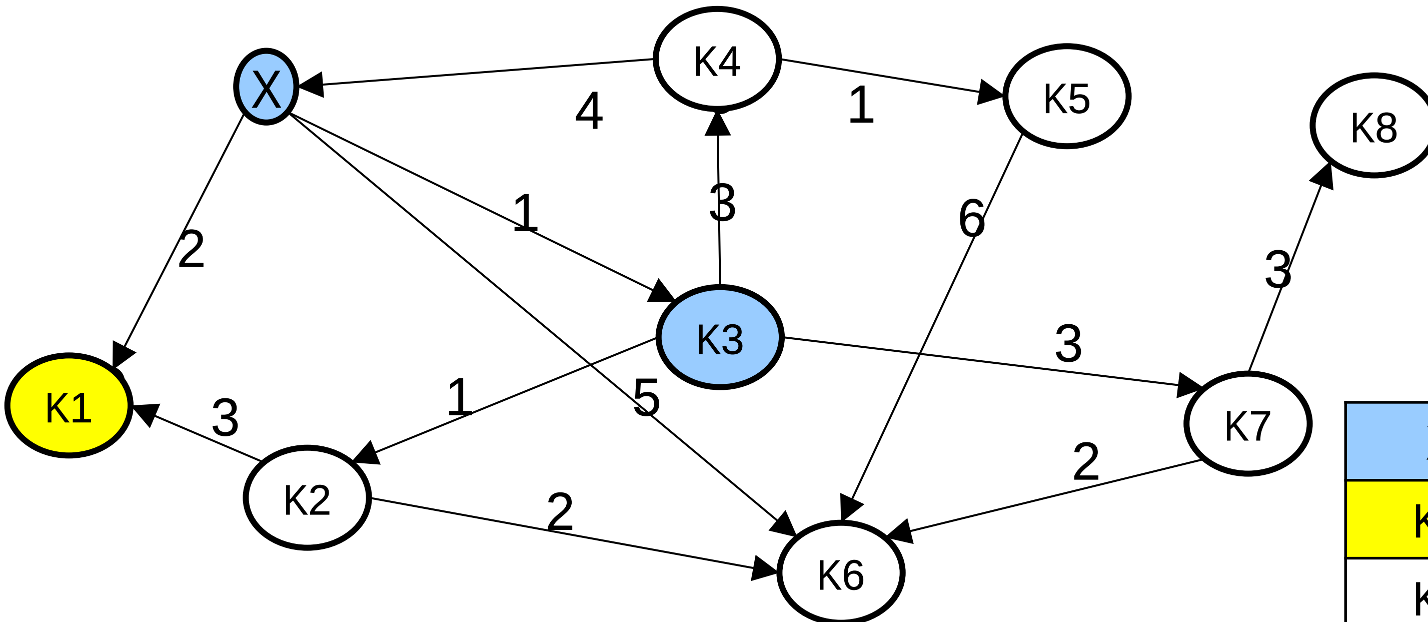
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	5
K7	4
K8	∞

- K3 is done (we cannot anymore find a shorter path)
- Pick K1 (or K2)

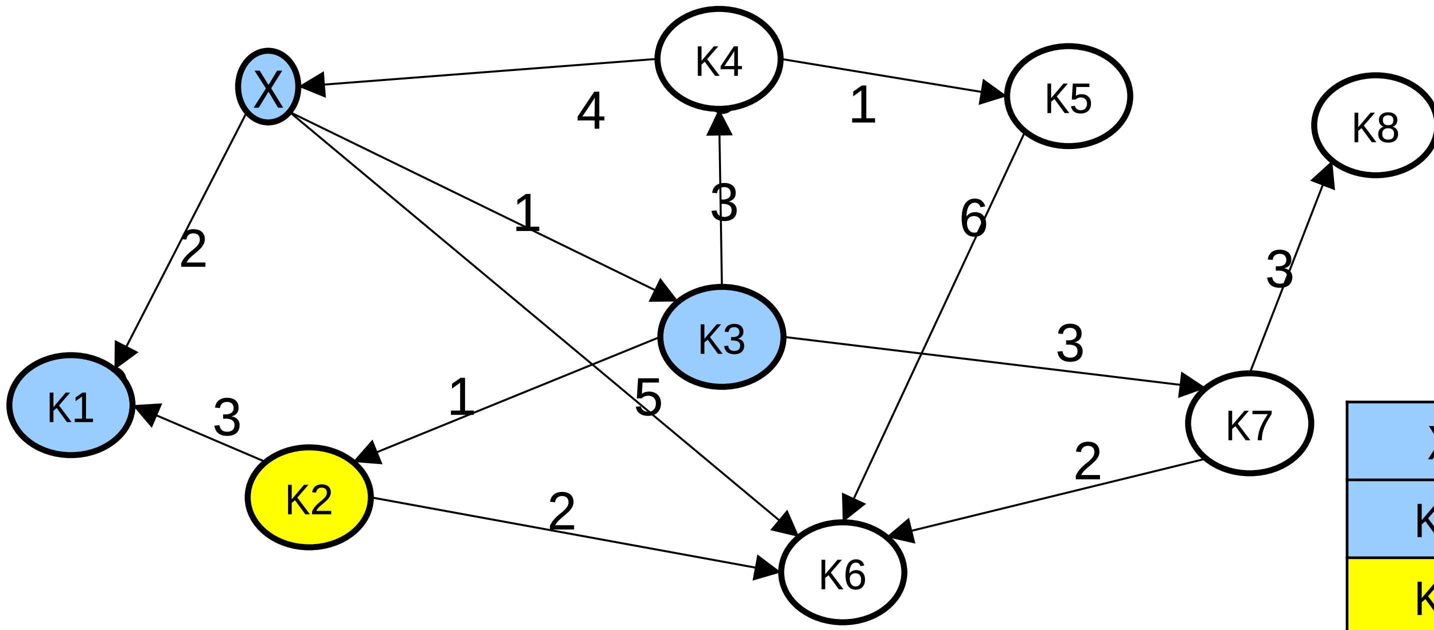
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	5
K7	4
K8	∞

- Pick K1
- Adapt distances to all neighbors
  - There are none

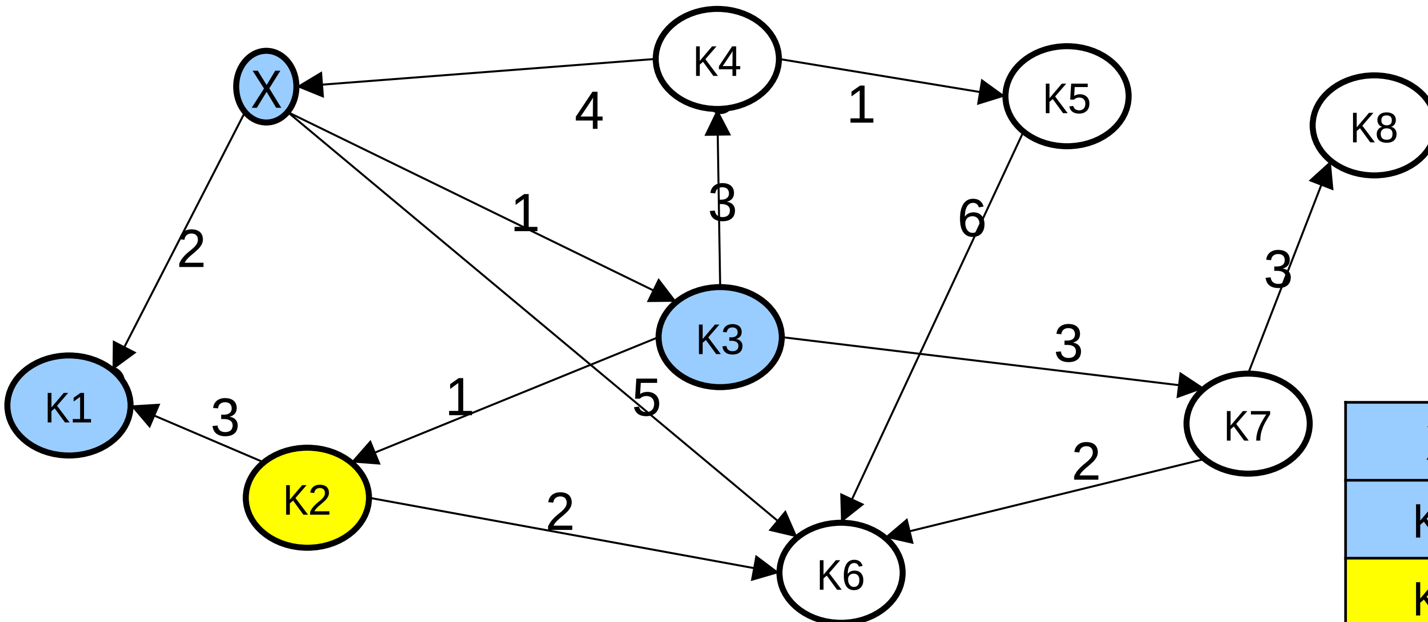
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	5
K7	4
K8	∞

- K1 is done
- Pick K2

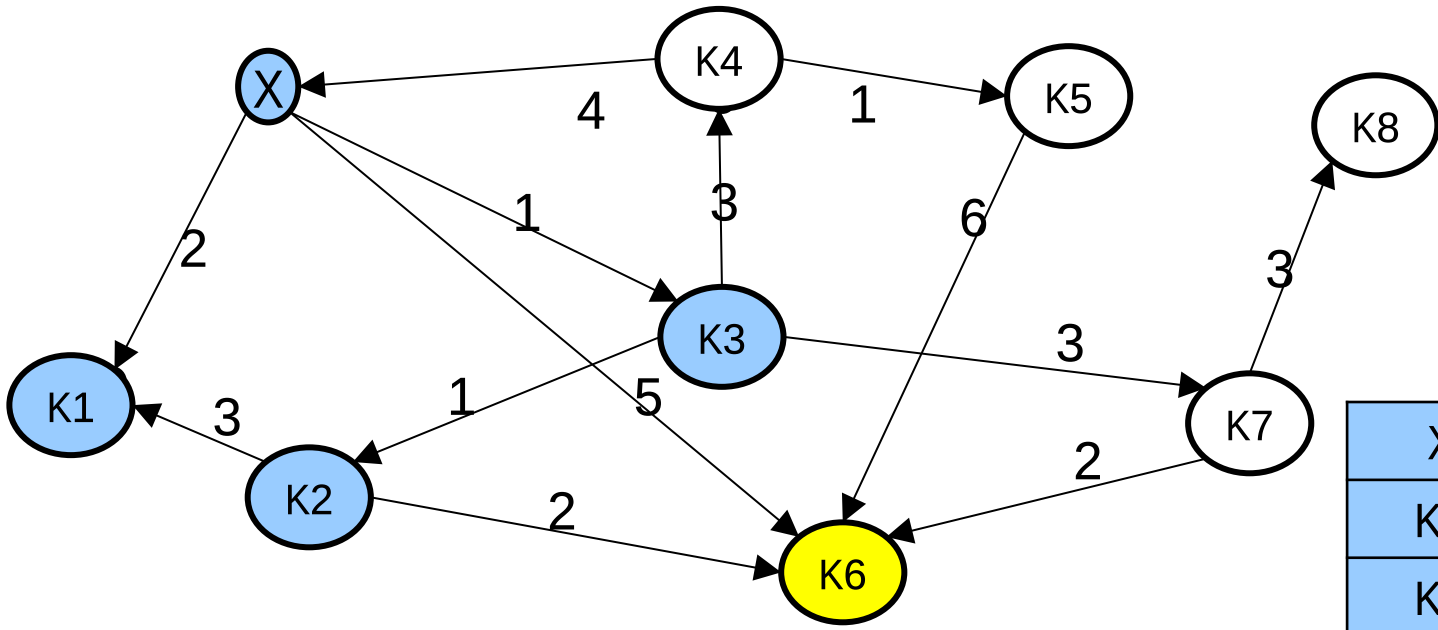
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	4
K7	4
K8	∞

- Pick K2
- Adapt distances to all neighbors
  - K1 was picked already – ignore
  - We found a shorter path to K6

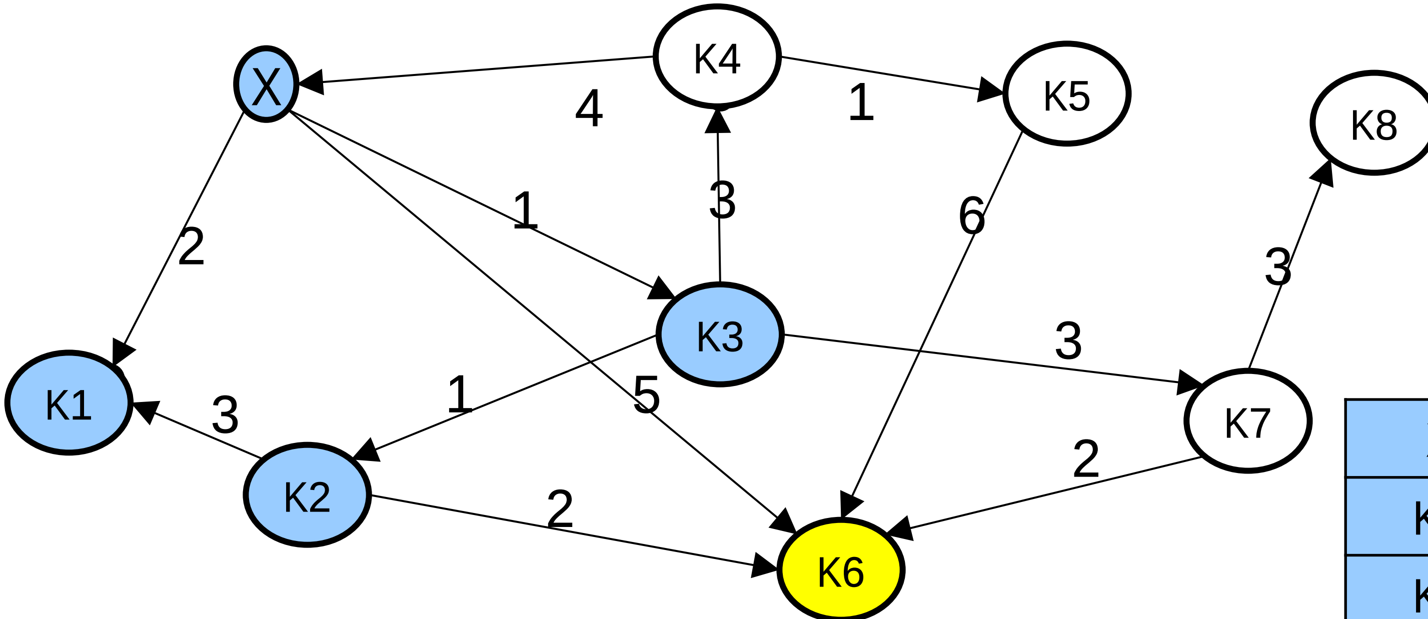
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	4
K7	4
K8	∞

- Pick K6 (or K4 or K7)

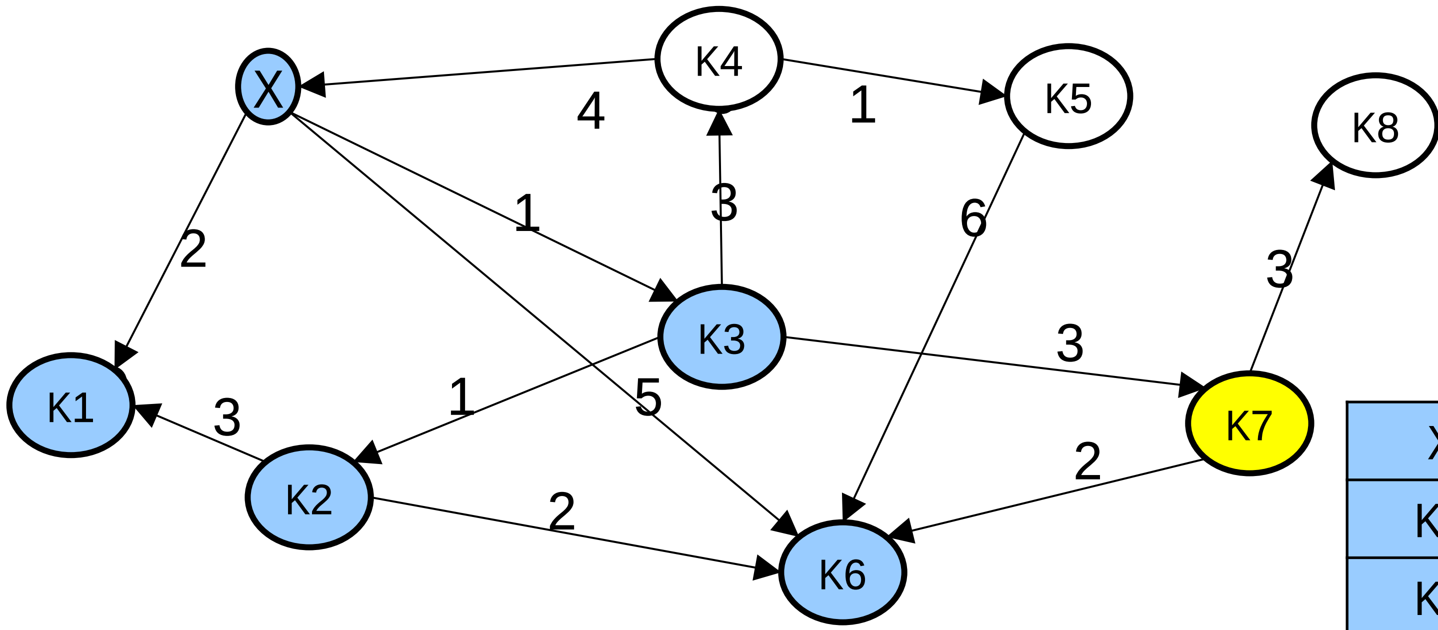
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	4
K7	4
K8	∞

- Pick K6
- Adapt distances to all neighbors
  – There are none

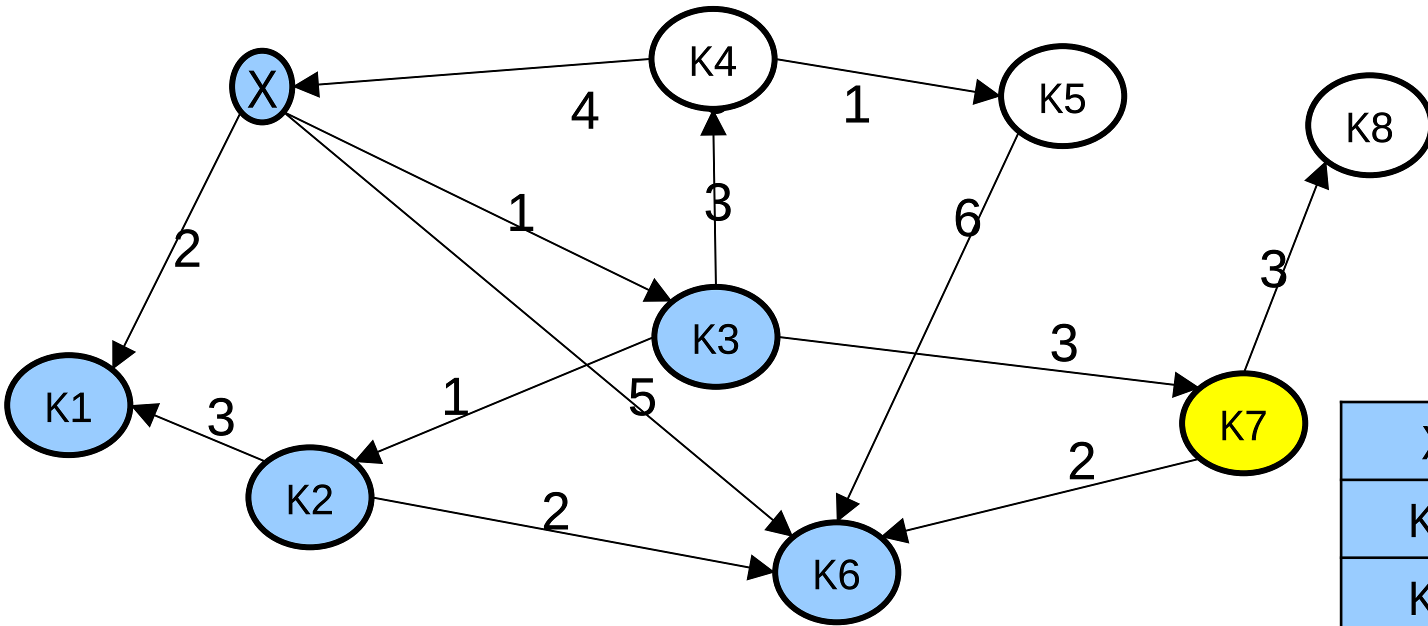
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	4
K7	4
K8	∞

- Pick K7

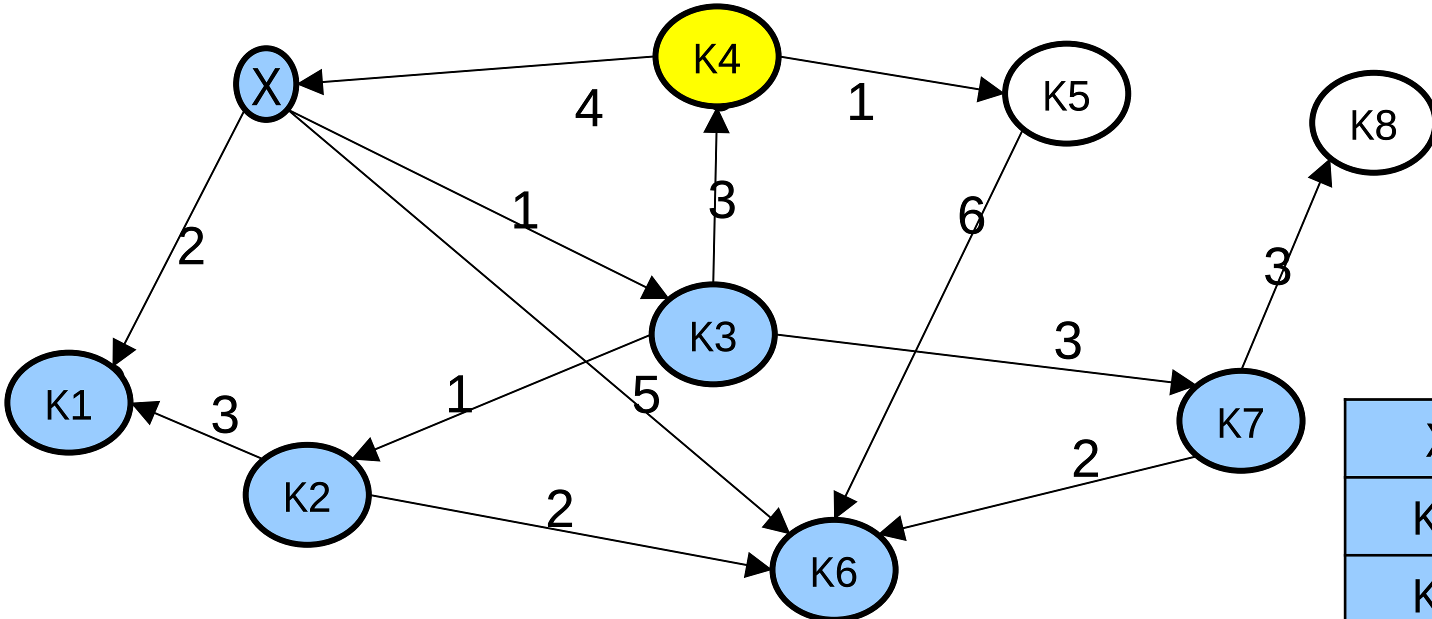
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	4
K7	4
K8	7

- Pick K7
- Adapt distances to all neighbors
  - K6 was visited already

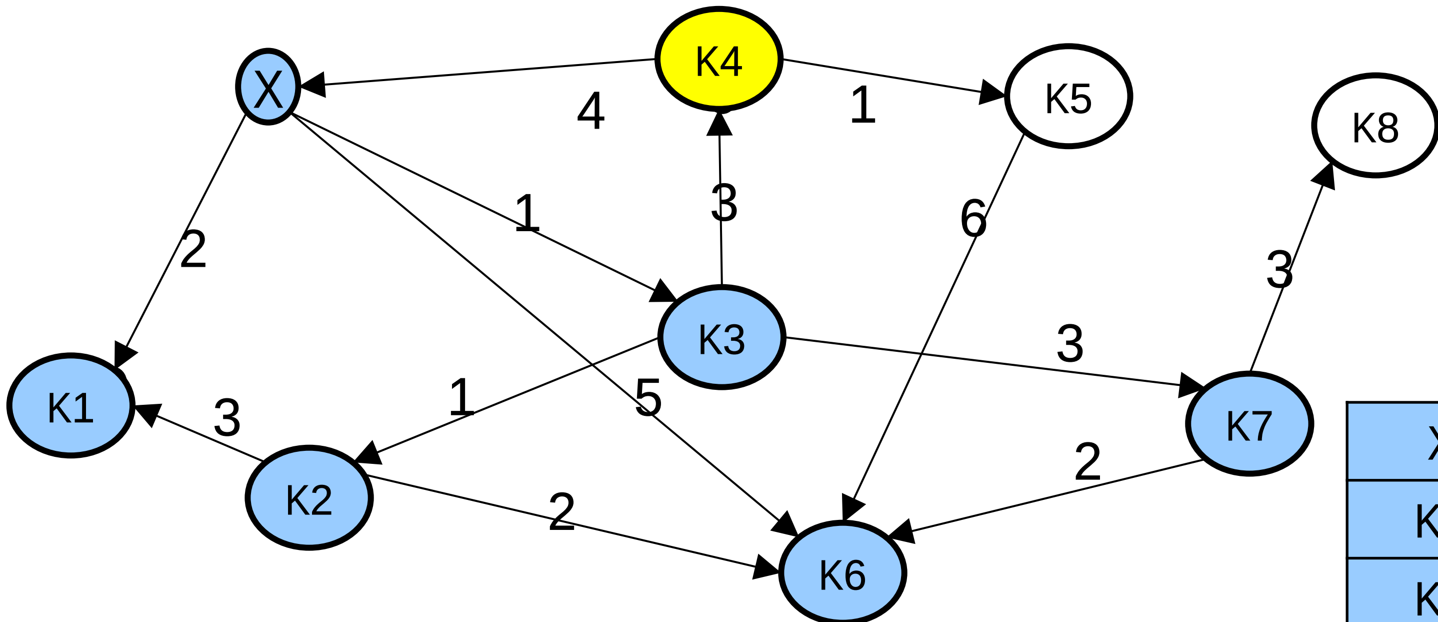
Example for Algorithm



- Pick K4

X	0
K1	2
K2	2
K3	1
K4	4
K5	∞
K6	4
K7	4
K8	7

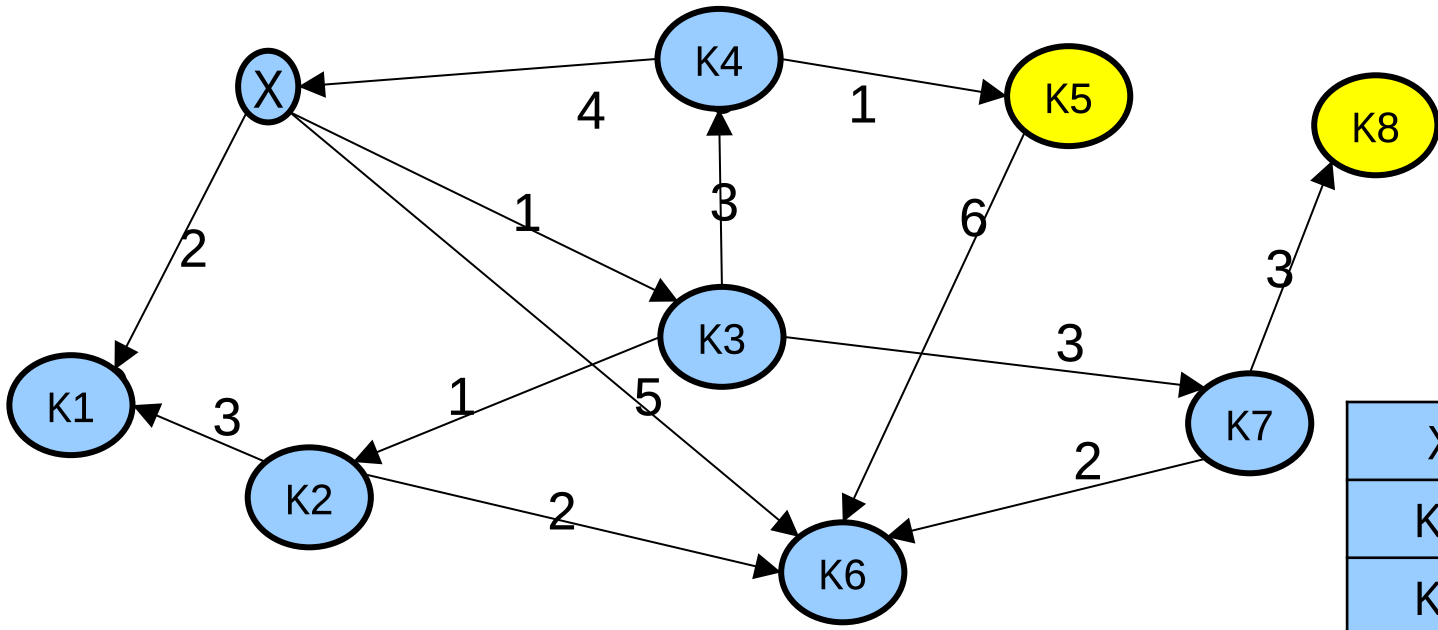
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	5
K6	4
K7	4
K8	7

- Pick K4
- Adapt distances to all neighbors
  - X was visited already

Example for Algorithm



- Pick K5 ... Pick K8

X	0
K1	2
K2	2
K3	1
K4	4
K5	5
K6	4
K7	4
K8	7

A Closer Look

```
G = (V, E);
x : start_node;    # x∈V
A : array_of_distances;
∀i: A[i]:= ∞;
L := V;
A[x] := 0;
while L≠∅
  k := L.get_closest_node();
  L := L \ k;
  forall (k,f,w)∈E do
    if f∈L then
      new_dist := A[k]+w;
      if new_dist < A[f] then
        A[f] := new_dist;
      end if;
    end if;
  end for;
end while;
```

- Algorithm seems to work
  - Proof and analysis will follow later
- Central: **`get_closest_node()`**
  - Needs to find the node k in L for which A[k] is the smallest
  - A[k] may change all the time
- Searching A? Resorting A?
- Better: Organize L as priority queue
  - List of tuples (o, v) (object,value)
  - All additions and updates of v
  - Make get_closest_node as fast as possible

Content of this Lecture

- Priority Queues
- Using Heaps
- Using Fibonacci Heaps

Priority Queues

- A priority queue (PQ) is an ADT with 3 essential operations
  - **`add( o,v)`**: Add element o with value (priority) v
  - **`getMin():`** Retrieve element with highest priority
  - **`removeMin():`** Remove element with highest priority
- Typical additional operations
  - **`merge( p1, p2):`** Merge two PQs into one
  - **`create( L)`**: Convert a list in a priority queue
  - **`delete( o):`** Delete o from PQ
  - **`changeValue( o, v):`** Change value of o to v

Other Applications

- Games (e.g. chess)
  - The machine explores next movements but cannot look at all of them; give each move an assumed benefit and explore moves with probably highest benefit first (see also A* algorithm)
- Multi-modal route planning
  - Find fastest route through a map (network) with multiple ways of transportation (feet, bus, train, ...) between edges where edge weights change dynamically (delay, congestion, ...)
    - And departure times may depend on arrival: Timetable-based routing
- Quality of Service in a network
  - When bandwidth is limited, sort all transmission requests in a PQ and transmit by highest priority
- ...

Naive Implementations (with |Q|=n)

- Using a linked list
  - **`add`** requires O(1) (at the end or start or anywhere)
  - **`getMin`** requires O(n) (bad)
  - **`deleteMin`** requires O(1) (if we keep the pointer after a getMin())
  - **`update`** requires O(n) (first search object)
  - **`merge`** requires O(1)
- Using a sorted linked list (by value/priority)
  - **`add`** requires O(n) (bad)
  - **`getMin`** requires O(1) (always first element)
  - **`deleteMin`** requires O(1)
  - **`update`** requires O(n) (search object, move to new position)
  - **`merge`** requires O(n+m)

Maybe Arrays?

- Using a sorted array
  - **`add`** requires O(n) (bad - we find the position in log(n), but then have to free a cell by moving all elements after this cell)
  - **`getMin`** requires O(1)
  - **`deleteMin`** requires O(n) (bad)
- PQs are typically used in applications where elements are inserted and removed (and updated) all the time
- We need a DS that can change its size dynamically at very low cost while keeping a certain order (min element)
- We want constant or at most log-time for all operations

Content of this Lecture

- Priority Queues
- Using Heaps
  - Heaps
  - Operations on Heaps
  - Heap Sort
- Using Fibonacci Heaps

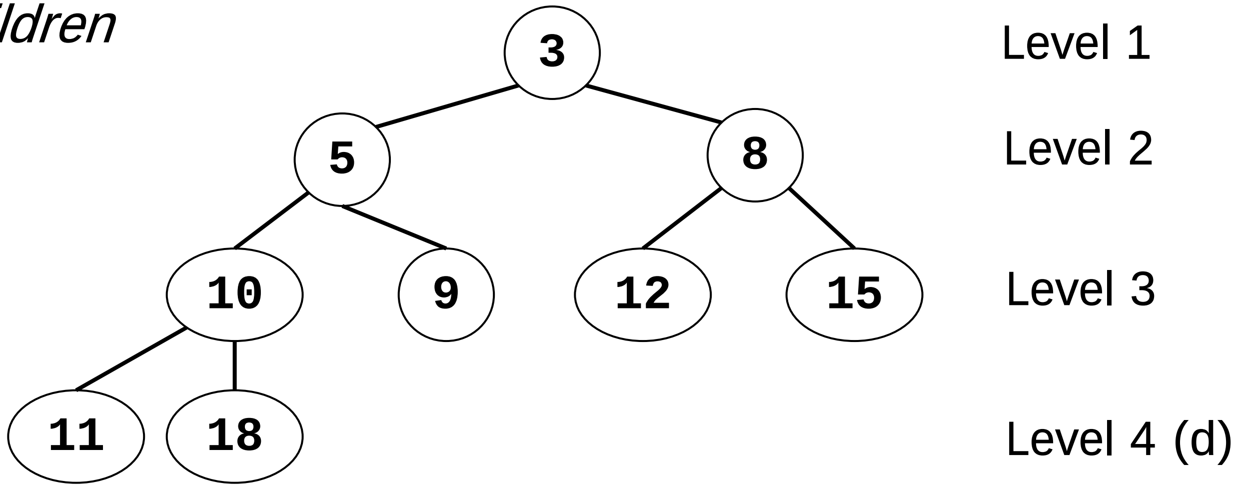
Heap-based PQ

- Unsorted lists require O(n) for **`getMin`**
  – We don‘t know where the smallest element is
- Sorted lists require O(n) for **`add`**
  – We don‘t know where to put the new element
- Can we find a way to keep the list “a little sorted”?
  – Actually, we only need the smallest element at a fixed position
  – All other elements can be at arbitrary places
  – Maybe **`add/deleteMin`** could be faster than O(n), if they don’t need to keep the entire list sorted
- One such structure is called a heap

Heaps

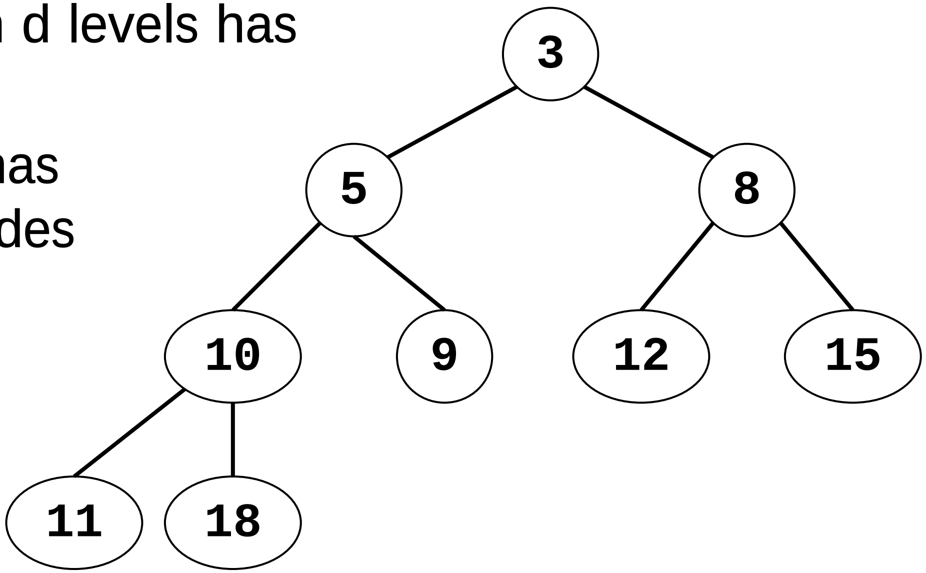
- Definition

  A heap is a labeled binary tree of depth d for which the following holds
  - *Nodes are labeled with integers (the priorities)*
  - *Form-constraint (FC): The tree is complete except the last level*
    - *I.e.: Every node at level l<d-1 has exactly two children*
  - *Heap-constraint (HC): The label of node is smaller than that of all its children*



Properties

- Order
  - A heap is “a little” sorted: We know the smallest element (root)
  - We know the order for some pairs of elements (parent-successors), but for many pairs we don’t know which is bigger
    - E.g. nodes in the same level
- Size
  - A complete binary tree with d levels has 2^d-1 nodes
  - A heap with m levels thus has between 2^{d-1}-1 and 2^d-1 nodes
  - A heap with n nodes has ceil(log(n+1)) levels

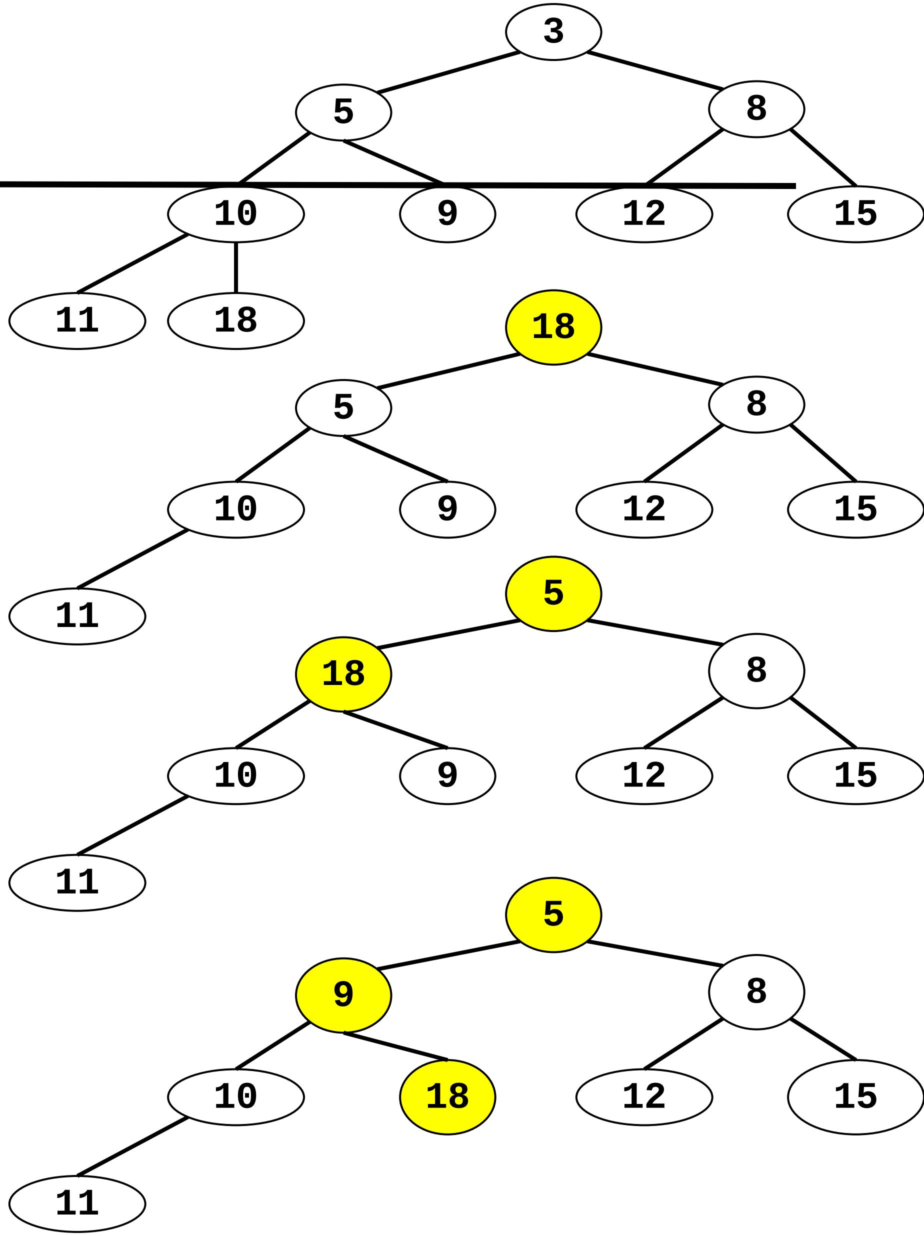


Operations

- Assume we store our PQ as a heap
- Clearly, **`getMin()`** is possible in O(1)
  - Keep a pointer to the root
- But ...
  - How can we cheaply perform **`deleteMin()`** – such that the new structure again is a heap?
  - How can we cheaply add an element to a heap – such that the new structure again is a heap?
  - How can we cheaply create a list – by turning a given list into a heap?

DeleteMin()

- We first remove the root
  - Creates two heaps
  - We must connect them again
- We take the „last“ node, place it in root, and “sift” it down the tree
  - Last node: right-most in the last level (actually, we can take any from the last level)
  - Sifting down: Exchange with smaller of both children as long as at least one child is smaller than the node itself



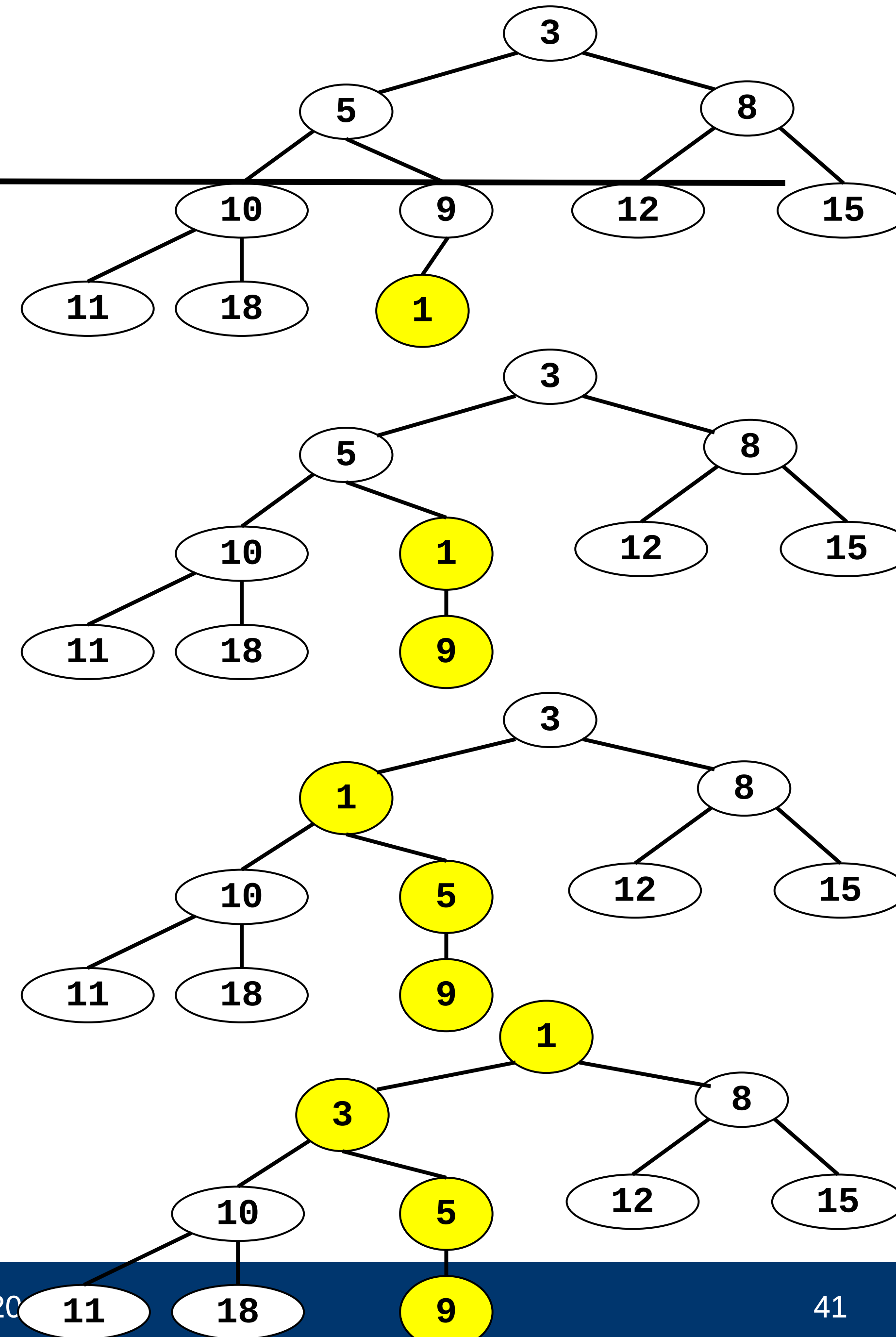
Analysis - Correctness

- We need to show that FC and HC still hold
- HC: Look at the tree after we sifted a node k. k may
  - ... be smaller than its children. Then HC holds and we are done
  - ... be larger than at least one child k2. Assume that k2 is the smaller of the two children (k1, k2) of k. We next swap k and k2. The new parent (k2) now is smaller than its children (k1, k), so the HC holds
  - After the last swap, k has no children – HC holds and we are done
- FC: We remove one node, then we sift down
  - Removing last node doesn't affect FC as we remove in the last level
  - Sifting does not change the topology of the tree (we only swap)

Analysis - Complexity

- Recall that a heap with n nodes has ceil(log(n+1)) levels
- During sifting, we perform at most one comparison and one swap in every level
- Thus: O(ceil(log(n+1))) = O(log(n))

Add() on a Heap



- Cannot simply add on top
- Idea: We add new element somewhere in last level and sift up
  - We might need a new level
  - Sifting up: Compare to parent and swap if parent is larger

Analysis

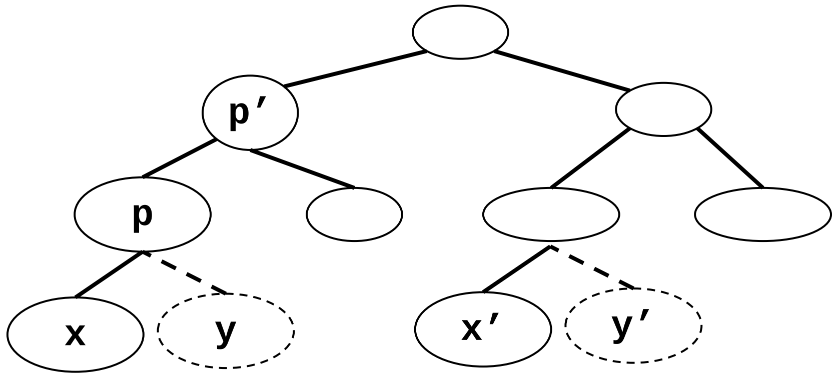
- Correctness
  - HC
    - If parent has only one child, HC holds after each swap
    - Assume a parent k has children k1 and k2, k2 was swapped there in the last move, and k2<k. Since HC held before, k<k1, thus k2<k<k1. We swap k2 and k, and thus the new parent is smaller than its children. On the other hand, if k2≥k, HC holds immediately (and we don’t swap).
  - FC: See **`deleteMin()`**
- Complexity: O(log(n))
  - See **`deleteMin()`**

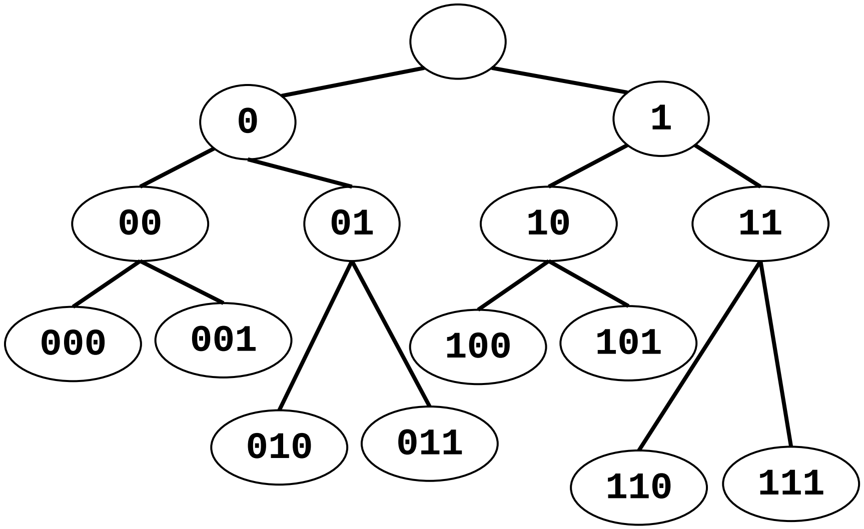
How to Find the Next Free / Last Occupied Node

- What do we need to find?
  - For **`deleteMin`**, we use the right-most leaf on the last level
  - For **`add`**, we add the leaf right from the last leaf
- We actually need the parent k
  - From |Q|=n, we can compute in O(1) the index p of the last leaf in the last level: p = n – 2^(floor(log(n)))
    - Or log(n+1) for **`add`**
  - The parent k of the node at p has index floor(p/2)'th in level d-1
  - The parent k' of k has index floor(p/4)'th in level d-2
  - ...
  - Now, in each node, we can decide whether to go left or right
  - Fast trick: Use the binary representation of p

Illustration

- For **`deleteMin`**, we need x (or x'); for **`add`**, we need y (or y')
  - p(x)=0, p(y)=1, p(x')=4, p(y')=5
  - Binary: 000, 001, 100, 101
- Go through bitstring from left-to-right
- Next bit=0: Go left
- Next bit=1: Go right
- Allows finding k in O(log(n))





Summary

	Linked list	Sorted linked list	Heap
getMin()	O(n)	O(1)	O(1)
deleteMin()	O(1)	O(1)	O(log(n))
add()	O(1)	O(n)	O(log(n))
merge()	O(1)	O(n1+n2)	O(log(n1)*log(n2))
Space	n add. pointer	n add. pointer	n add. pointer

Heaps can be kept efficiently in an array – no extra space, but limit to heap size

Creating a Heap

- We start with an unsorted list with n elements
- Naïve algorithm: Start with empty heap and perform n additions
  - Obviously requires O(n*log(n))
- Better: Bottom-Up-Sift-Down
  - Build a tree from the n elements fulfilling the FC (but not HC)
    - Simple fill a tree level-by-level – this is in O(n)
  - Sift-down all nodes on the second-last level
  - Sift-down all nodes on the third-last level
  - ...
  - Sift down root

Analysis

- Correctness
  - After finishing one level, all subtrees starting in this level are heaps because sifting-down ensures FC and HC (see **`deleteMin()`**)
  - Thus, when we are done with the first level (root), we have a heap
- Analysis
  - We look at the cost per level h (1 ... log(n)=d)
  - For every node at level h, we need at most d-h operations
  - At level h≠d, there are 2^{h-1} nodes
    - For nodes at level d, we don't do anything
  - Over all levels, this yields

$$T(n) = \sum_{h=1}^{d-1} 2^{h-1} * (d-h) = \sum_{h=1}^{d-1} h * 2^{d-h-1} = 2^{d-1} \sum_{h=1}^{d-1} \frac{h}{2^h} \le n * \sum_{h=1}^{\infty} \frac{h}{2^h} = n * 2 = O(n)$$

Side Note: Heap Sort

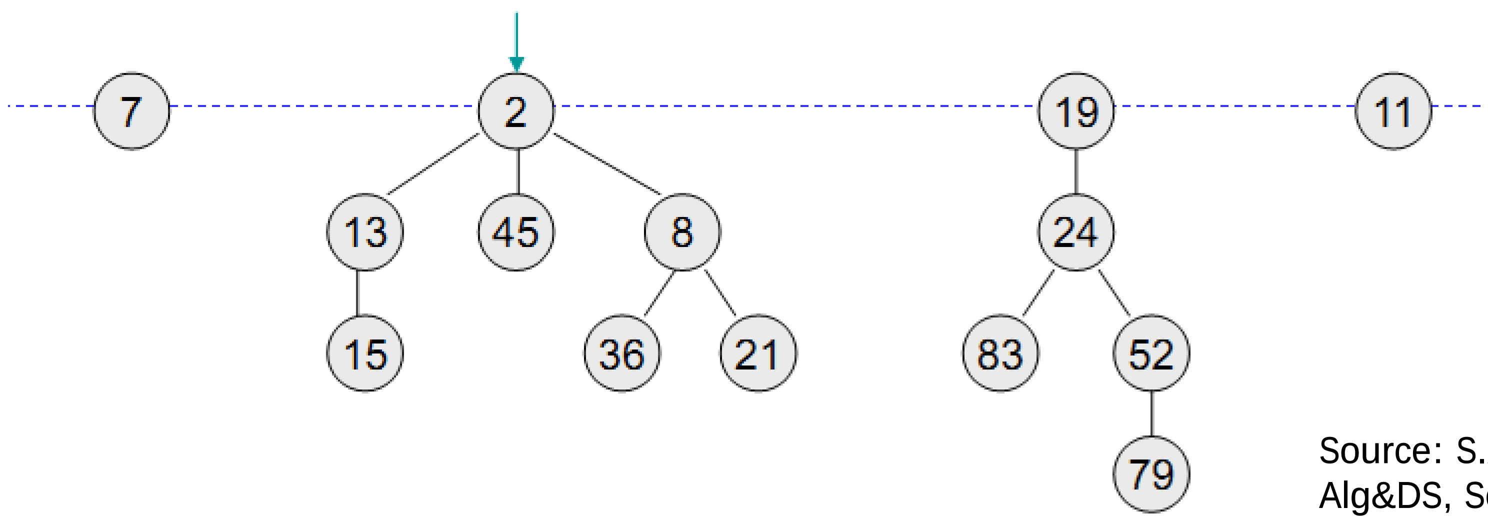
- Heaps also are a suitable data structure for sorting
- Heap-Sort (a classical sorting algorithm)
  - Given an unsorted list, first create a heap in O(n)
  - Repeat
    - Take the smallest element and store in array in O(1)
    - Re-build heap in O(log(n))
      - Call **`deleteMin( root)`**
  - Until heap is empty – after n iterations
- Thus: O(n*log(n))
  - Average-case only slightly better
- Can be implemented in-place when heap is stored in array
  - See [OW93] for details

Content of this Lecture

- Priority Queues
- Using Heaps
- Using Fibonacci Heaps

Fibonacci-Heaps (very rough sketch)

- A Fibonacci Heap (FH) is a forest of (non-binary) heaps with disjoint values
  - All roots are maintained in a double-linked list
  - Special pointer (**`min`**) to the smallest root
  - Accessing this value (**`getMin()`**) obviously is O(1)



Source: S.Albers, Alg&DS, SoSe 2010

Maintainance of a FH

- FHs are maintained in a lazy fashion
  - `add(v)`: We create a new heap with a single element node with value v. Add this heap to the list of heaps; adapt min-pointer, if v is smaller than previous min
    - Clearly O(1)
  - merge(): Simple link the two root-lists and determine new min (as min of two mins)
    - Clearly O(1)
- Deleting an element (`deleteMin()`) needs more work
  - Until now, we just added single-element heaps
  - Thus, our structure after n add() is an unsorted list of n elements
  - Finding the next min element after `deleteMin()` in a naïve manner would require O(n)

deleteMin() on FH

- Method is not complicated
  - We first remove the min element
  - We then go through the root-list and merge heaps with the same rank (=# of children) until all heaps in the list have different ranks
  - Merging two heaps in O(1): (1) Find the heap with the smaller root value; (2) Add it as child to the root of the other heap
- But analysis is fairly complicated
  - The above method is O(n) in worst case
    - But after every clean-up, the root-list is much smaller than before
    - Subsequent clean-ups need much less time
  - Amortized analysis shows: Average-case complexity is O(log(n))
  - Analysis depends on the growth of the trees during merge – these grow as the Fibonacci numbers

Disadvantage

- Though faster on average, Fibonacci Heaps have unpredictable delays
- No log(n) upper bound for every operation
- Not suitable for real-time applications etc.

Summary

	Linked list	Sorted linked list	Heap	Fibonacci Heap
getMin()	O(n)	O(1)	O(1)	O(1)
deleteMin()	O(1)	O(n)	O(log(n))	O(log(n))*
add()	O(1)	O(n)	O(log(n))	O(1)
merge()	O(1)	O(n1+n2)	O(log(n))	O(1)

*: Amortized analysis