Algorithms and Data Structures

Sorting beyond Value Comparisons

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Content of this Lecture

- Radix Exchange Sort
  - Sorting bitstrings in linear time (almost)
- Bucket Sort (aka LSD Radix Sort)

Knowledge

- Until now, we did not use any knowledge on the nature of the values we sort
  - Strings, integers, reals, names, dates, revenues, person's age
  - Only operation we used: "value1 < value2"
  - Exception: Our (refused) suggestion (max-min)/2 for selecting the pivot element in Quicksort (how can we do this for strings?)
- Use knowledge on data type: Positive integers
- First example
  - Assume a list S of n different integers, $\forall i: 1 \leq S[i] \leq n$
  - How can we sort S in O(n) time and with only n extra space?

Sorting Permutations

```
1. S: array_permuted_numbs;
2. B: array_of_size_|S|
3. for i:= 1 to |S| do
4.   B[S[i]] := S[i];
5. end for;
```

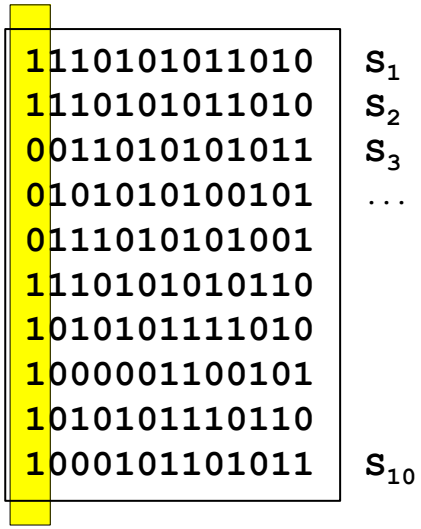
- Very simple
  - If we have all integers [1, n], then the final position of value i must be i
  - Obviously, we need only one scan and only one extra array (B)
- Knowledge we exploited
  - There are n different, unique values
  - The set is „dense" – no value between 1 and n is missing
  - It follows that the position of a value in the sorted list can be derived from the value

Removing Pre-Requisites

- Assume S is not dense (no duplicates)
  - n different integers each between 1 and m with m>n
  - For a given value S[i], we do not know any more its target position
    - How many values are smaller?
    - At most min(S[i], n)
    - At least max(n-(m-S[i]), 0)
  - This is almost the usual sorting problem, and we cannot do much
- Assume S has duplicates
  - S contains n values, where each value is between 1 and m and appears in S with m<n
  - Again: We cannot directly infer the position of S[i] from i alone

Second Example: Sorting Binary Strings

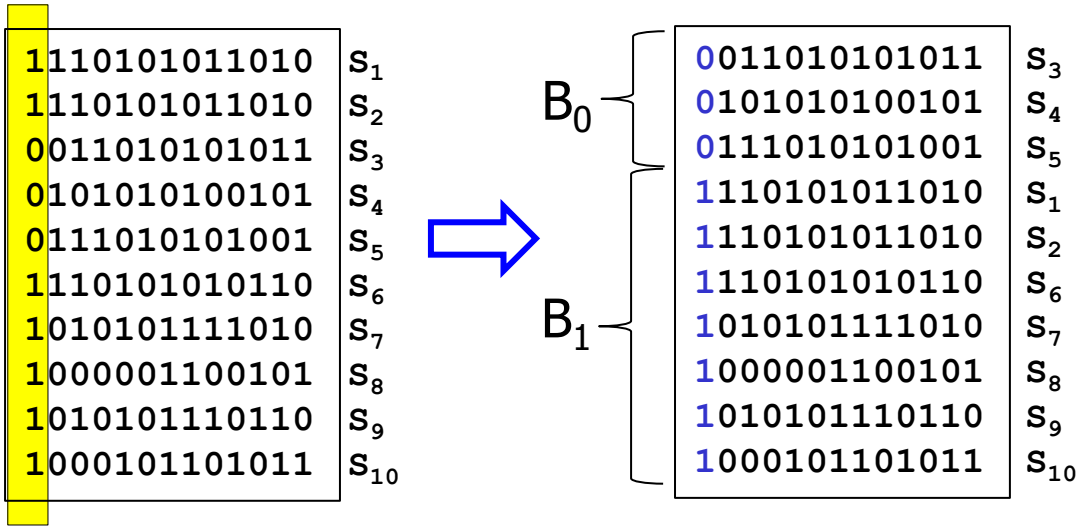
- Assume that all values are binary strings (bistrings) of equal length
  - E.g., integers in machine representation
- The most important position is the left-most bit, and it can have only two different values
  - Alphabet size is 2 in bitstrings



Second Example: Sorting Binary Strings

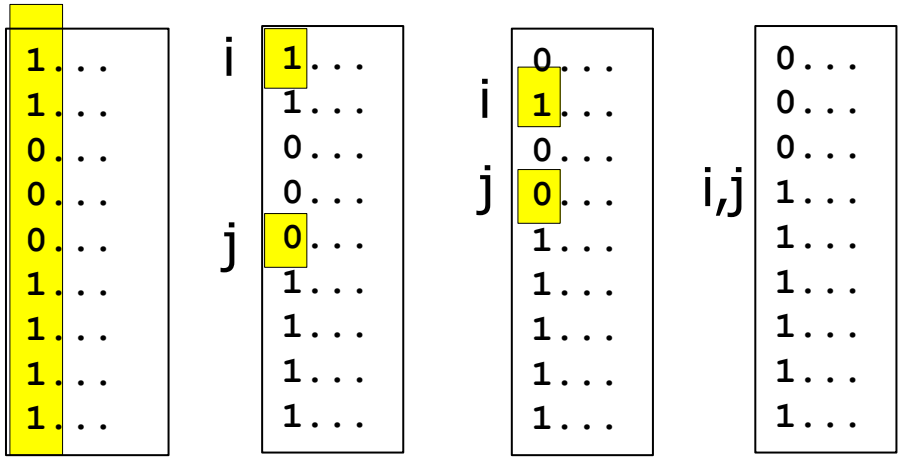
- We can sort all values by first position with a single scan
  - All values with leading 0 => list B0
  - All values with leading 1 => list B1

```
1.  S: array_bitstrings;
2.  B0: array_of_size_|S|
3.  B1: array_of_size_|S|
4.  j0 := 1;
5.  j1 := 1;
6.  for i:= 1 to |S| do
7.    if S[i][1]=0 then
8.      B0[j0] := S[i];
9.      j0 := j0 + 1;
10.   else
11.     B1[j1] := S[i];
12.     j1 := j1 + 1;
13.   end if;
14. end for;
15. return B0[1..j0]+B1[1..j1];
```



Improvement

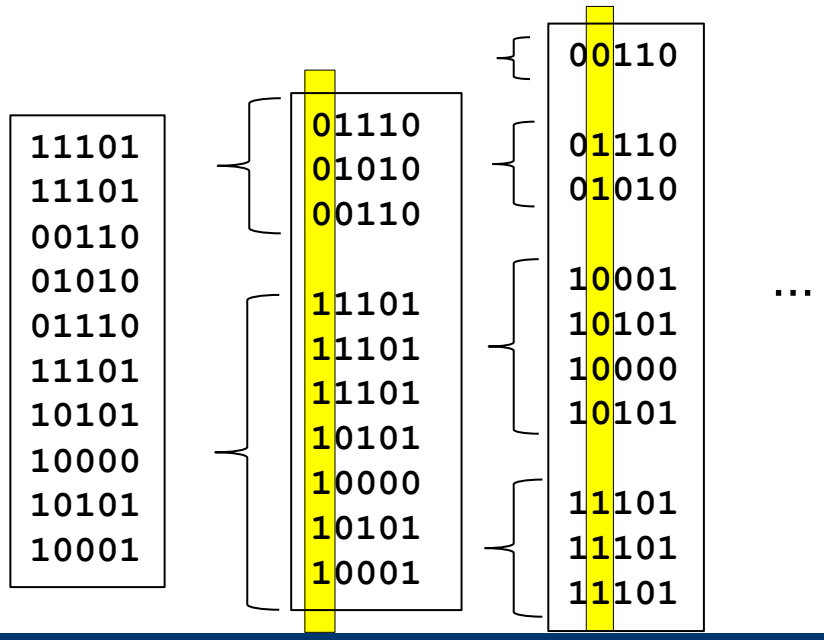
- How can we do this in O(1) additional space?
- Recall QuickSort
  - Call **`divide*(S,1,1,|S|)`**
  - k, l, r, and return value will be used in a minute
  - Note that we return j, the position of the first 1




```
1.  func int divide*(S array;
2.                   k,l,r: int) {
3.    i := l;
4.    j := r;
5.    repeat
6.      while S[i][k]=0 and i<j do
7.        i := i+1;
8.      end while;
9.      while S[j][k]=1 and i<j do
10.       j := j-1;
11.     end while;
12.     swap(S[i], S[j]);
13.   until i=j;
14.   if S[r][k]=0 then //only zeros
15.     j:=j+1;
16.   end if
17.   return j;    // first "1"
18. }
```

Sorting Complete Binary Strings

```
1.  func radixESort(S array;
2.                  k,l,r: integer) {
3.    if l≥r or k>m then
4.      return;
5.    end if;
6.    d := divide*(S, k, l, r);
7.    radixESort(S, k+1, l, d-1);
8.    radixESort(S, k+1, d, r);
9.  }
```



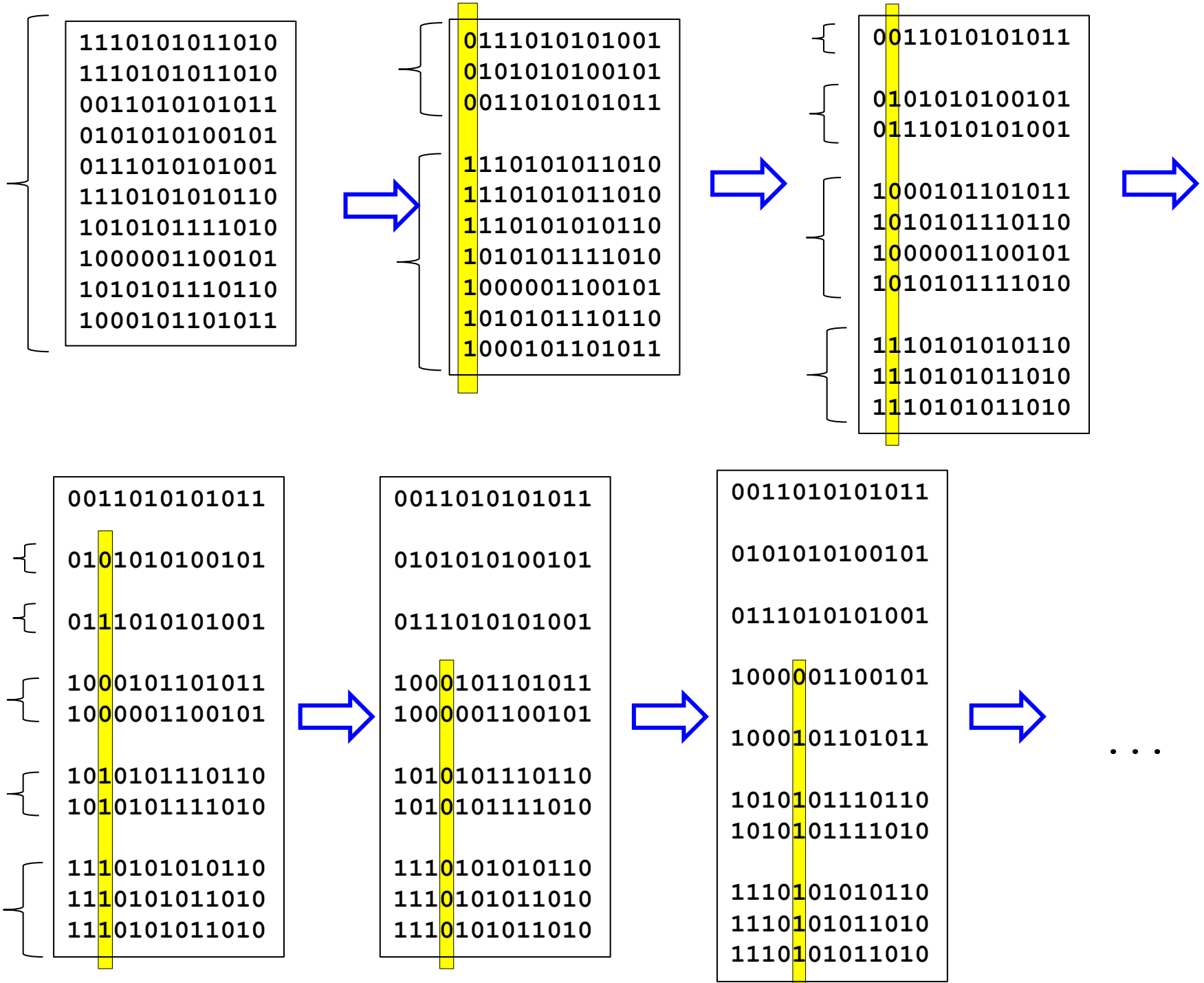
- We can repeat the same procedure on the second, third, … position
- When sorting the k'th position, we need to take care to not sort the entire S again, but only the subarrays with same values in the (k-1) first positions
  - Let m by the length (in bits) of the values in S
  - Call with `radixESort(S,1,1,|S|)`

Complexity

```
1.  func radixESort(S array;
2.                  k,l,r: integer) {
3.    if l≥r or k>m then
4.      return;
5.    end if;
6.    d := divide*(S, k, l, r);
7.    radixESort(S, k+1, l, d-1);
8.    radixESort(S, k+1, d, r);
9.  }
```

```
1.  func int divide*(S array;
2.                   k,l,r: int) {
3.    …
4.    repeat
5.      while S[i][k]=0 and i<j do
6.        i := i+1;
7.      end while;
8.      while S[j][k]=1 and i<j do
9.        j := j-1;
10.     end while;
11.     swap(S[i], S[j]);
12.   until i=j;
13.   …
14.   return j;   // first "1" }
```

- We count the overall number of comparisons for...
  - In divide*, we look at every element S[l…r] exactly once: (r-l)
  - Then we divide S[l…r] in two disjoint halves
    - 1st makes (d-l) comps
    - 2nd makes (r-d) comps
  - The first call to radixESort has O(n) comps, with |S|=n.
- Are we in O(n)?


1110101011010
1110101011010
0011010101011
0101010100101
0111010101001
1110101010110
1010101111010
1000001100101
1010101110110
1000101101011
0111010101001
0101010100101
0011010101011
1110101011010
1110101011010
1110101010110
1010101111010
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0111010101001
1000001100101
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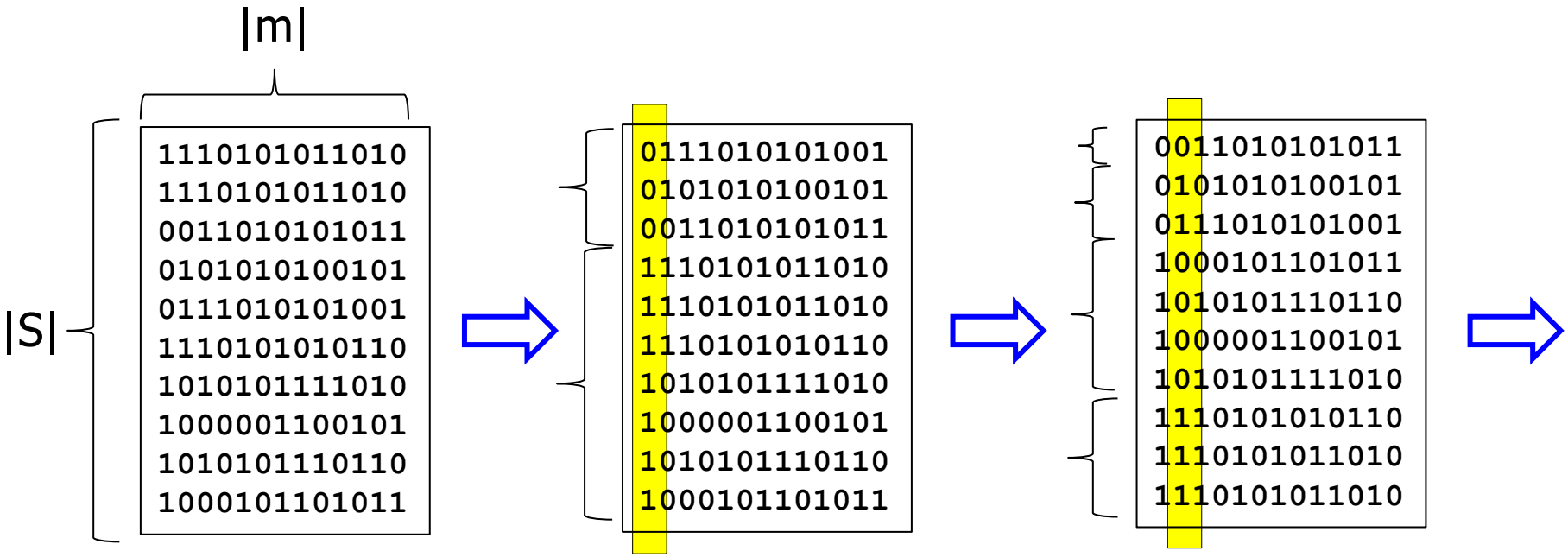
Illustration



- For every k, we look at every S[i][k] once to see whether it is 0 or 1 – together, we have at most m*|S| comparisons
  - Of course, we can stop at every interval with (r-l)=1
  - m*|S| is the worst case

Complexity (Correct)

```
1.  func radixESort(S array;
2.                  k,l,r: integer) {
3.    if l≥r or k>m then
4.      return;
5.    end if;
6.    d := divide*(S, k, l, r);
7.    radixESort(S, k+1, l, d-1);
8.    radixESort(S, k+1, d, r);
9.  }
```

```
1.  func int divide*(S array;
2.                   k,l,r: int) {
3.    …
4.    repeat
5.      while S[i][k]=0 and i<j do
6.        i := i+1;
7.      end while;
8.      while S[j][k]=1 and i<j do
9.        j := j-1;
10.     end while;
11.     swap(S[i], S[j]);
12.   until i=j;
13.   …
14.   return j;   // first "1" }
```

- We count …
  - Every call to radixESort first performs (r-l) comps and then divides S[l…r] in two disjoint halves
    - 1st makes (d-l) comps
    - 2nd makes (r-d) comps
- First call to radixESort has O(n) comps, with |S|=n
- Recursion depth is fixed to m
- Thus: O(m*|S|) comps

RadixESort - Bonuses

- It may not have to examine all the bits/positions

```
00110101010111   = 2 / 13
01010101001011   = 3 / 13
01110101010011   = 3 / 13
10000011001011   = 5 / 13
10001011010111   = 5 / 13
10101011101110   = 11 / 13
10101011110110   = 11 / 13
11101010101110   = 10 / 13
11101010110110   = 13 / 13
11101010110110   = 13 / 13
```

~58% of bits examined

- It works for variable length bitstrings (equal bits have to be at equal positions or padded with 0)

```
00110101
0101010100101
01110
10000011
10001011
10101011101
10101011110
11101010101
```

RadixESort or QuickSort?

- Assume we have data that can be represented as bitstrings such that more important bits are left (or right – but consistent)
  - Integers, strings, bitstrings, …
  - Equal length is not necessary, but „the same" bits must be at the same position in the bitstring (otherwise, one may pad with 0)
- Decisive: m <? >? log(n)
  - If S is large / maximal bitstring length is small: RadixESort
  - If S is small / maximal bitstring length is large: QuickSort

Content of this Lecture

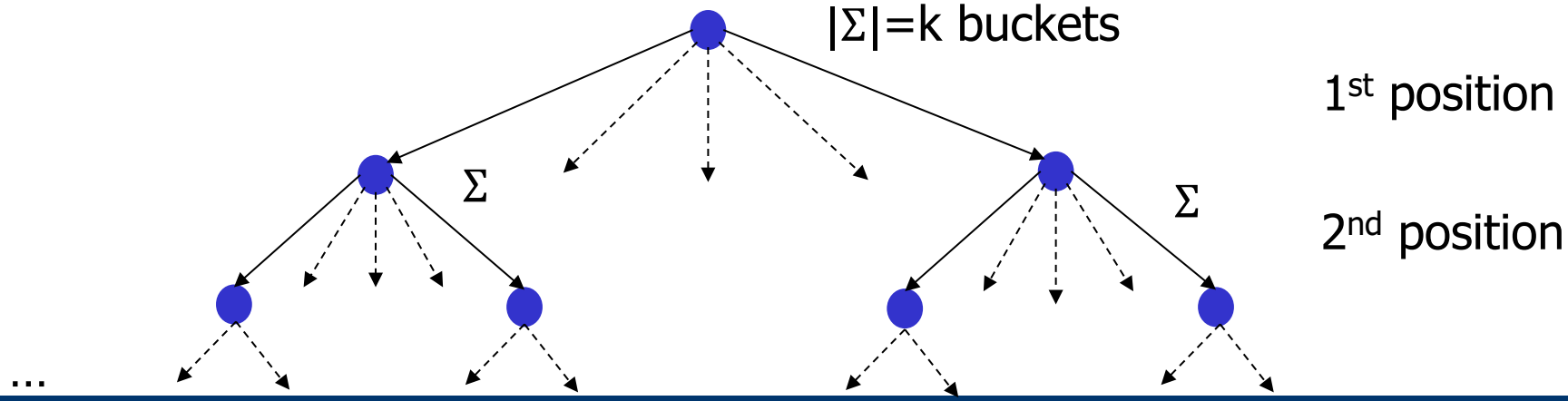
- Radix Exchange Sort
- Bucket Sort
  - Generalizing the Idea of Radix Exchange Sort to arbitrary alphabets

Bucket Sort

- Representing "normal" Strings as bitstrings is a bad idea
  - One byte per character -> 8*length bits (large m for RadixESort)
  - But: There are only ~26 different values (no case)
- One could find shorter encodings – we go a different way

Bucket Sort generalizes RadixESort

- Assume |S|=n, m being the length of the largest value, alphabet Σ with |Σ|=k and a lexicographical order (e.g., "A" < "AA")
- We first sort S on first position into k buckets (with a single scan)
- Then sort every bucket again for second position, etc.
- After at most m iterations, we are done
- Time complexity (ignoring space issues): O(m*(|S|+…))
- But space is an issue



Space in Bucket Sort

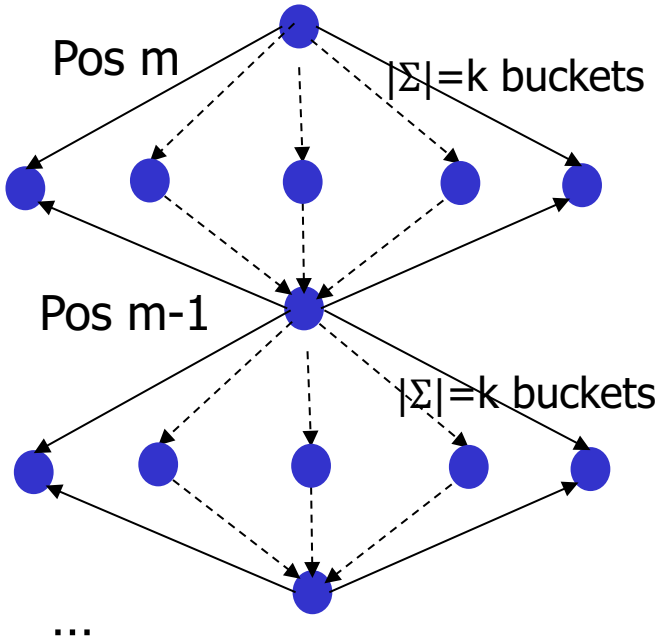
- A naïve array-based implementation allocates $k * |S|$ values for every phase of sorting into each bucket B
  - We do not know how many values start with a given character
  - Can be anything between 0 and |S|
- This would need to allocate a total of $O(k^m * |S|)$ space for m iterations – too much!

- We reduce this to $O(k + |S|)$
  - Requires a stable sorting algorithm for single characters
  - 1-phase Bucket Sort is stable (if implemented that way)

Bucket Sort - Idea

- If we sort from back-to-front and keep the order of once sorted suffixes, we can (re-)use the additional space
  - Order was not preserved in RadixESort, but there we could sort in-place – other problems

```
1.  func bucketSort(S array, m,k: integer){
2.    B:= Array of Queues with |B|=k
3.    for i := m down to 1 do
4.       use stable sorting algorithm
         to sort S on position I
5.       merge all buckets
6.    end for
7.  }
```



Bucket Sort

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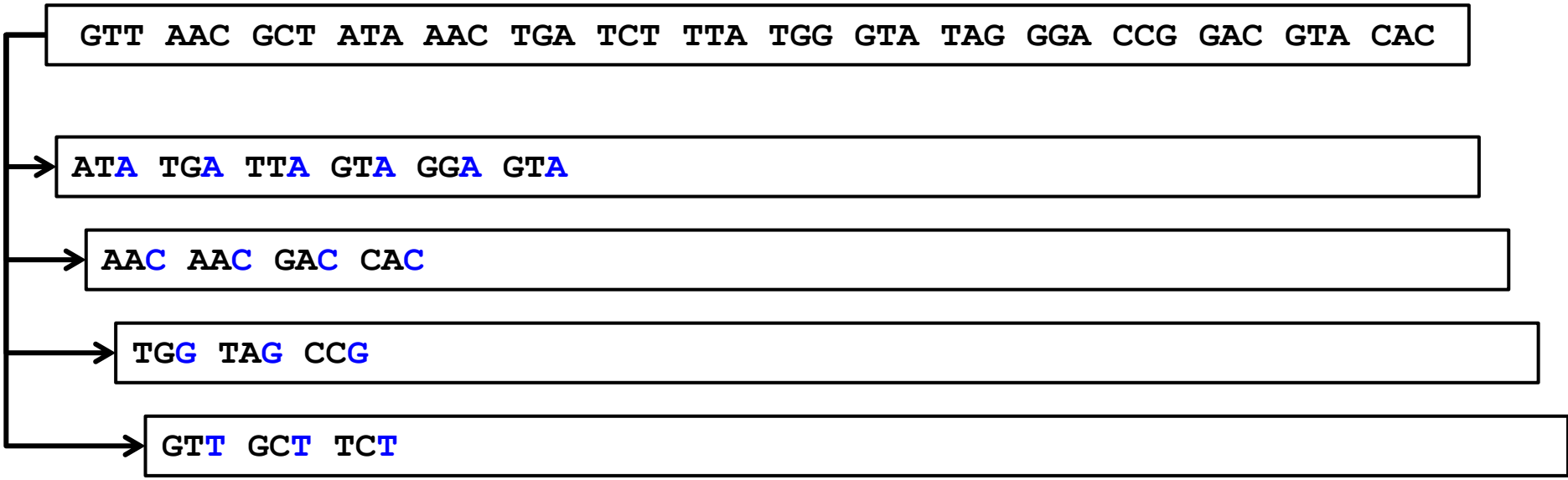

Bucket Sort

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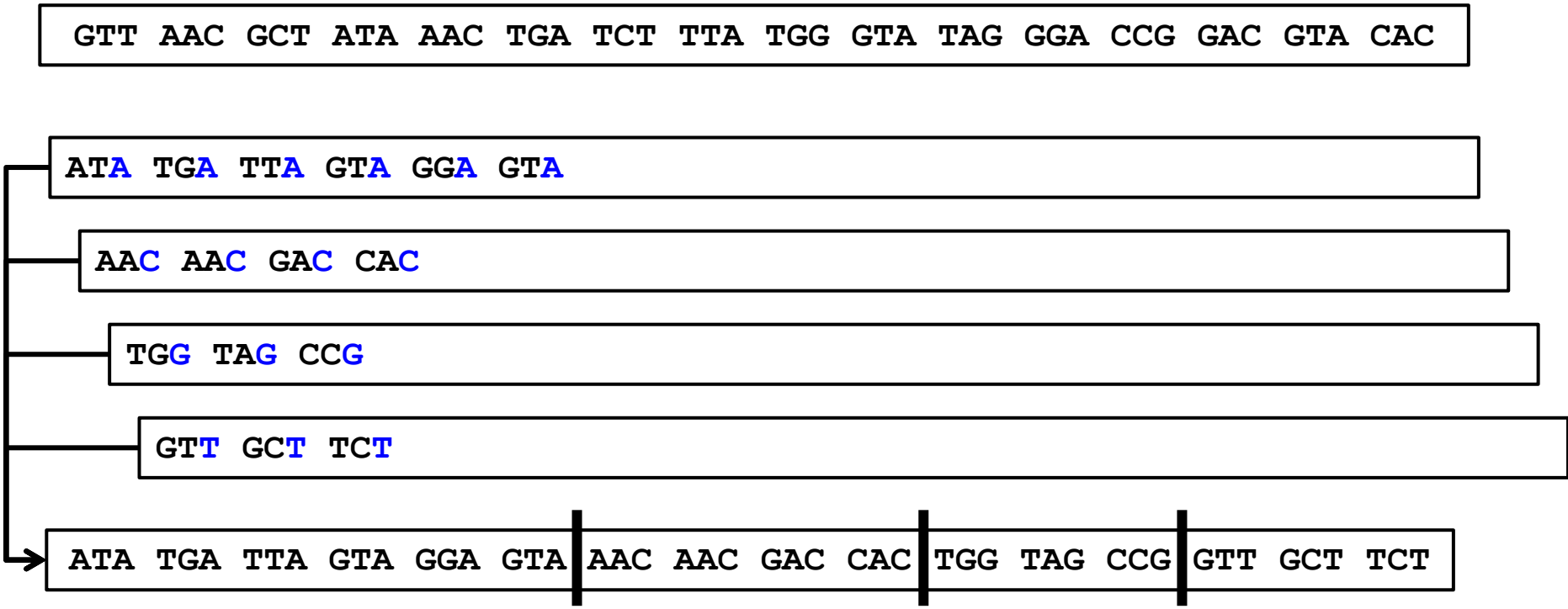
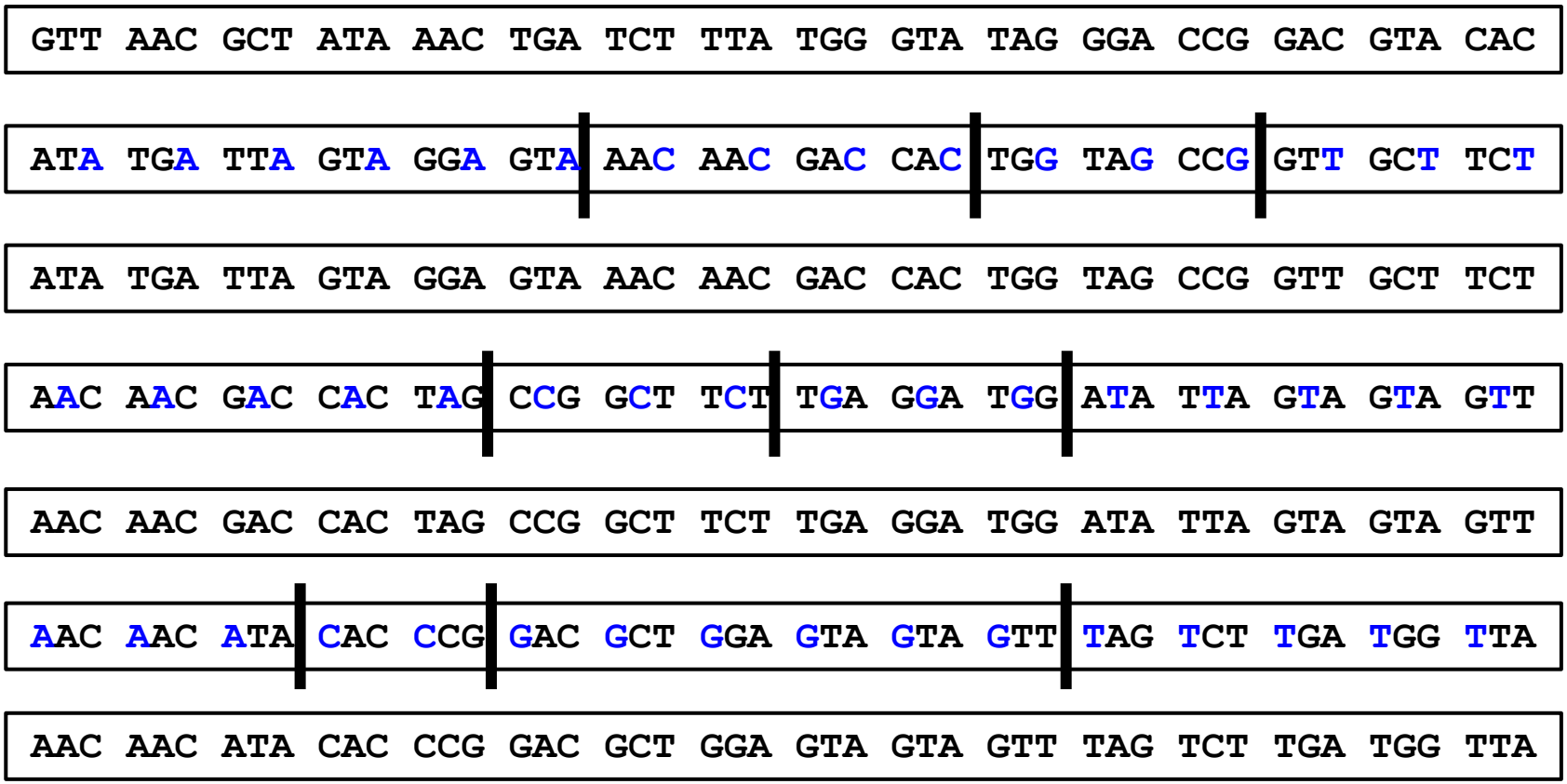


Bucket Sort

- If we sort from back-to-front and keep the order of once sorted suffixes, we can (re-)use the additional space



Bucket Sort – Pseudocode (aka LSD Radix Sort)

- Sort S from back-to-front
  - (Re-)use k queues, one for each bucket.
  - *findBucket* translates the i-th char of S[j] into a bucket.
  - Stable: the order of once sorted suffixes is kept (append to end of queue).
  - Finally, merge buckets and repeat with next position.

- Different mappings can be used, such as ‚A-Z' mapped to 1-26.
  - For linear time sorting, the number of buckets must be equal to k.

```
1.  func bucketSort(S array,
                    m, k: integer){
2.    B:= Array of Queues with |B|=k
3.    for i := m down to 1 do
4.      for j := 1 to |S| do
5.        k := findBucket(S[j][i]);
6.        B[k].enqueue(S[j]);
7.      end for
8.      j := 1;
9.      for k := 1 to |B| do
10.       while not B[k].isEmpty() do
11.         S[j] := B[k].dequeue();
12.         j := j + 1;
13.       end while end for
14.   end for
15.   return S;
16. }
```

Magic? Proof

- By induction
- Assume that before phase t we have sorted all values by the (t-1)-suffix (right-most, least important for order)
  - True for t=2 – we sorted by the last character ((t-1)-suffixes)
- In phase t, we sort by the t‘th lowest value (from the right)
- This will group all values from S with the same value in S[i][m-t+1] together and keep them sorted wrt. (t-1)-suffixes
  - Assuming a stable sorting algorithm
- Since we sort by S[i][m-t+1], the array after phase t will be sorted by the t-suffix
- qed.

Saving More Space

- The example has shown that we actually never need more than $O(|S| + k)$ additional space (all buckets together)
  - Use a linked-list/queue for each bucket
  - Keep pointer to start (for copying) and end (for extending) of each list – this requires 2*k space
  - All lists together only store |S| elements (of length m)

A Word on Names

- Names of these algorithms are not consistent
  - Radix Sort generally depicts the class of sorting algorithms which look at single keys and partition keys in smaller parts
  - RadixESort is also called binary quicksort (Sedgewick)
  - Bucket Sort is also called „Sortieren durch Fachverteilen“ (OW), RadixSort (German WikiPedia and Cormen et al.), or LSD Radix Sort (Sedgewick), or distribution sort
  - Cormen et al. use Bucket Sort for a variation of our Bucket Sort (linear only if keys are equally distributed)
  - …

Summary

	Comps worst case	avg. case	best case	Additional space	Moves (wc / ac)
Selection Sort	$O(n^2)$		$O(n^2)$	O(1)	O(n)
Insertion Sort	$O(n^2)$		O(n)	O(1)	$O(n^2)$
Bubble Sort	$O(n^2)$		O(n)	O(1)	$O(n^2)$
Merge Sort	O(n*log(n))		O(n*log(n))	O(n)	O(n*log(n))
QuickSort	$O(n^2)$	O(n*log(n)	O(n*log(n)	O(log(n))	$O(n^2)$ / O(n*log(n))
BucketSort (m= …)	O(m*(n+k))			O(n+k)	

Exemplary Questions

- What is the best case complexity of BucketSort?
- What is the space complexity of RadixESort?
- What is a stable sorting algorithm?
- Which of the following sorting algorithms are stable: BubbleSort, InsertionSort, MergeSort?
- BucketSort needs a data structure for building and using buckets. Give an implementation using (a) a heap, (b) a queue.