Comparing Axiomatizations of Free Pseudospaces

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Abstract

Independently and pursuing different aims, Hrushovski and Srour [HS89] and Baudisch and Pillay [BP00] have introduced two free pseudospaces that generalize the well know concept of Lachlan's free pseudoplane. In this paper we investigate the relationship between these free pseudospaces, proving in particular, that the pseudospace of Baudisch and Pillay is a reduct of the pseudospace of Hrushovski and Srour.

1 Introduction

Already back in 1974 Lachlan [Lac74] introduced the free pseudoplane which is by now a well studied and well understood model-theoretic object. In particular, Hrushovski and Pillay [HP85] showed that 1-based or weakly normal theories do not contain a type-definable pseudoplane. Hence the free pseudoplane is the prototype of a stable and not 1-based theory.

While the free pseudoplane is a 2-dimensional object in essence, two generalizations of the pseudoplane in form of 3-dimensional pseudospaces were independently introduced by Hrushovski and Srour [HS89] and Baudisch and Pillay [BP00]. The motivations for the construction of these pseudospaces differ, but the constructions itself share many common features. On the other hand, the axiomatizations are at first sight of a comparatively different style, with even differently chosen language for the two pseudospaces. It is the main purpose of this paper to clarify the relationship between these two pseudospaces. In particular, we construct a standard model of the free pseudospace of Hrushovski and Srour and prove that the free pseudospace of Baudisch and Pillay is a reduct of this pseudospace. This relationship was already conjectured in [BP00] but the actual verification is far from being obvious.

The free pseudospace of Hrushovski and Srour is the first example of a stable and non-equational theory. Equational theories were introduced by Srour [PS84, Sro88] and further developed by Junker and Kraus [Jun00, JK02]. A formula $\varphi(\bar{x}, \bar{y})$ is called an equation, if every intersection $\bigcap_{i \in I} \varphi(\bar{x}, \bar{a}_i)$ of instances of φ is equivalent to a sub-intersection $\bigcap_{i \in I_0} \varphi(\bar{x}, \bar{a}_i)$ with finite $I_0 \subseteq I$. A theory is equational, if every formula is equivalent to a Boolean combination of equations.

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By counting the number of types it is easy to see that equational theories are stable [PS84]. Thus Srour posed the question whether the class of equational theories is a proper subclass of the class of stable theories. This question was answered affirmatively by Hrushovski and Srour with the construction of their free pseudospace in the unfortunately unpublished manuscript [HS89]. The result from [HP85] mentioned above shows that Lachlan's pseudoplane is a typical example of a stable non-1-based theory. As equational theories provide a natural generalization of 1-based theories [PS84], this motivates the approach to search for a stable non-equational theory in form of a higher-dimensional version of the pseudoplane.

Independently of [HS89], Baudisch and Pillay [BP00] constructed another free pseudospace as an example of a non-CM-trivial stable theory in which no infinite field is interpretable. This shows that the hierarchy of n-ample theories, developed by Pillay [Pil00], is strict up to its second level. The first level of this hierarchy is again formed by non-1-based theories, whereas 2-ample theories correspond to non-CM-trivial theories.

This paper is organized as follows. In Sect. 2 we review the pseudoplane of Lachlan [Lac74]. We also introduce a colored version of this pseudoplane which will serve as an essential ingredient for the analysis of the pseudospace of Hrushovski and Srour.

In Sects. 3 and 4 we describe the free pseudospaces Σ of Baudisch and Pillay [BP00] and Γ of Hrushovski and Srour [HS89]. Using the standard model of Σ from [BP00] we construct a standard model of Γ.

The main results follow in Sect. 5 where we investigate the relationship between the axiom systems Σ and Γ. We prove that Σ is a reduct of Γ. The main technical difficulty for this result lies in deriving from Γ the axioms of Σ which expresses the freeness conditions. We achieve this by analyzing paths and circles in models of Γ. As a byproduct we obtain a simplification of the axiom system Σ.

In the final section we explain the original purpose of Γ as a stable non-equational theory. In particular, we include a full proof for the non-equationality of Γ which is based on the proof sketch given in the draft [HS89].

2 The Free Pseudoplane

First we will review the free pseudoplane of Lachlan [Lac74], because it is of fundamental importance for the higher dimensional pseudospaces that are the topic of this paper. The language contains unary predicates B and C for lines and points, respectively, and a binary incidence relation I between lines and points. The free pseudoplane is axiomatized by the following axiom set Δ:

Δ1) Every element is a point or a line, but not both.

Δ2) $I \subseteq (B \times C) \cup (C \times B)$ is a symmetric relation between lines and points.

Δ3) Every point lies via I on infinitely many lines. Conversely, every line contains infinitely many points.

Δ4) There are no circles, i.e., there do not exist mutually distinct elements $x_0, \ldots, x_n$, $n \geq 2$, with $I(x_i, x_{i+1})$, $0 \leq i \leq n-1$, and $I(x_n, x_0)$.

The standard model N_0 of Δ has as its domain the set $\omega^{<\omega}$ of finite sequences of natural numbers. The lines of N_0 are the sequences of even length, whereas sequences of odd length are points. The incidence relation $I(x, y)$ holds between elements x and y, if x is either a direct predecessor or a direct successor of y. Thus N_0 is a countable model of Δ, which is moreover connected. It is well known that Δ is a complete theory.

Next we will describe a colored modification of the free pseudoplane, where lines and points are equipped with colors. This modification is not of independent interest, but it will serve as an important building block in subsequent sections. The language is enriched by unary relations C_r, C_w, B_r and B_w for red and white points and red and white lines, respectively. The axiom set Δ' contains in addition to the axioms Δ1) to Δ4) the following three axioms regarding the colors:

Δ5) Every line is either red or white, i.e., it fulfills exactly one of the predicates B_r or B_w. The similar condition holds for points.

Δ6) Every point lies on infinitely many white and on infinitely many red lines.

Δ7) Every red (resp. white) line b contains exactly one red (resp. white) point, which is called the *exceptional point* of b.

Models of Δ' are called *free colored pseudoplanes*. The standard model N_0' of the colored pseudoplane is derived from the standard model N_0 of Δ by coloring lines and points. Lines are colored according to

$$\begin{aligned} B_r(N_0') &= \{b \,|\, b \in B(N_0'),\ b_{\ell(b)} \text{ is even}\} \\ B_w(N_0') &= \{b \,|\, b \in B(N_0'),\ b_{\ell(b)} \text{ is odd}\}\ , \end{aligned}$$

where $\ell(b)$ denotes the length of the sequence b, and $b_{\ell(b)}$ is its last element. By this construction every point lies on infinitely many red and white lines.

It remains to color the points. If the predecessor point c of a line b in $B(N_0')$ has a different color than b, then c is the exceptional point of b, and all successors of b are colored with the color of b. If, on the other hand, b and c are of the same color, then we can choose the exceptional point freely among the successors of b. Therefore Δ7) is fulfilled, and hence N_0' is a model of Δ'.

It is not hard to directly construct an isomorphism between two countable connected free colored pseudoplanes. Therefore also the theory Δ' of the colored pseudoplane is complete.

3 The Free Pseudospace of Baudisch and Pillay

In this section we describe a 3-dimensional analogon of the pseudoplane as developed by Baudisch and Pillay [BP00]. In addition to points and lines the pseudospace contains also planes. The language L of this pseudospace consists of unary predicates A, B, C for planes, lines and points, respectively, and binary predicates I and J for the incidence relations between planes and lines as well as between lines and points.

Before we describe the axioms of the pseudospace we need to introduce some terminology. By A, B and C we also denote the set of planes, lines and points, respectively. We will usually use letters $a, a', a_i \dots$ for planes, $b, b', b_i \dots$ for lines and $c, c', c_i \dots$ for points, and we will often refrain from indicating explicitly the type of an element denoted in this way. Planes and lines are identified with the set of its points, i.e., $a = \{c \,|\, (\exists b) J(a, b) \text{ and } I(b, c)\}$ and $b = \{c \,|\, I(b, c)\}$. This allows the use of expressions like $c \in b$, $b \subset a$ or $a \cap b$, which are considered as abbreviations for the respective formulas involving the incidence relations I and J. Further, we define for a plane a the sets $B(a) = \{b \in B \,|\, J(a, b)\}$ and $C(a) = \{c \in C \,|\, c \in a\}$. For a point c the sets $A(c)$ and $B(c)$ are defined analogously.

Elements $d_0, \dots, d_n$ from a *sequence*, if $I(d_i, d_{i+1})$ or $J(d_i, d_{i+1})$ for all $0 \leq i \leq n-1$. If additionally for all $0 \leq i < j \leq n$ with $(i, j) \neq (0, n)$ we have $d_i \neq d_j$, then $(d_0, \dots, d_n)$ is called a *path*. If all elements d_i are planes or lines, then we speak of an AB-path. BC-paths are similarly defined. The length of a path is the number of different elements in it, i.e. the length of the above path is either n or $n + 1$. A *circle* is a path $(d_0, \dots, d_n)$ with $d_0 = d_n$.

The free pseudospace of [BP00] is axiomatized by the following axioms Σ:

$\Sigma 0)$ Every element fulfills exactly one of the relations A, B, or C. $J \subseteq (A \times B) \cup (B \times A)$ and $I \subseteq (B \times C) \cup (C \times B)$ are symmetric.

$\Sigma 1)(a)$ (A, B, J) is a free pseudoplane. Dually, we have:

$\Sigma 1)(b)$ (B, C, I) is a free pseudoplane.

$\Sigma 2)(a)$ $(B(a), C(a), I)$ is a free pseudoplane for every plane a, and dually:

$\Sigma 2)(b)$ $(A(c), B(c), J)$ is a free pseudoplane for every point c.

$\Sigma 3)(a)$ The intersection of two planes is either empty, or a point, or a line.

$\Sigma 3)(b)$ The set of planes that contain two distinct points is either empty, or exactly one plane, or the set of planes containing a common line.

$\Sigma 4)(a)$ Let a be a plane and $X = (a, b, \dots, b', a)$ be a circle of length n. Then there exists a BC-path between b and b' of length at most $n - 1$, which only contains points from X and lines from a.

$\Sigma 4)(b)$ Let c be a point and let $X = (c, b, \dots, b', c)$ be a circle of length n. Then there exists an AB-path of length at most $n - 1$, which contains only planes from X and lines from $B(c)$.

It is apparent from the axioms that points and planes are completely dual to each other. Many arguments can therefore be simplified by establishing some property only for points and lines, which immediately implies this property for planes and lines as well. In Sect. 5 we will prove that it is in fact not necessary to include this duality in the axiomatization. It already follows from the (a)-parts of the axioms of Σ.

In [BP00] Baudisch and Pillay construct a countable connected standard model M_0 of Σ. Further, it is shown that the theory Σ is complete, ω-stable, and not CM-trivial.

4 The Free Pseudospace of Hrushovski and Srour

This section is devoted to another free pseudospace, introduced by Hrushovski and Srour [HS89]. Although this pseudospace is very similar to the pseudospace of Baudisch and Pillay [BP00], it also contains a number of additional features. Before giving the full axiomatization we will provide an informal description.

As in the pseudospace of [BP00] models consist of points, lines, and planes. As before there are incidence relations I between points and lines and J between lines and planes, but additionally there are two direct incidence relations I_r and I_w between points and lines. Lines are either red or white, indicated by unary relations B_r and B_w. Points are also red or white, where the color of points is specified by the binary relations I_r and I_w between points and planes. In particular, points can change their color from plane to plane. Via I_r and I_w planes split into a red and a white section. A red line b of a plane a contains only points from the red section of a, except for one white point, the exceptional point of b in a. The same holds for white lines. Lines and points of a plane therefore form a free colored pseudoplane. Finally, there are axioms stating that models are maximally free of circles.

The language L' consists of unary relation symbols A, B, B_r, B_w, and C for planes, lines (red and white) and points, and binary relation symbols I, J, I_r, and I_w for the incidence relations. Therefore L' extends the language L from the previous section. The axiom set Γ from [HS89] contains the following axioms:

Γ0) Every element fulfills exactly one of the relations A, B, or C. Lines are either red or white, i.e., every line fulfills exactly one of the relations B_r or B_w. $I \subset (B \times C) \cup (C \times B)$ is symmetric.

Γ1) $I_r, I_w \subset (A \times C) \cup (C \times A)$ are the symmetric incidence relations between planes and white and red points, respectively. $I_r \cap I_w = \emptyset$. The red and white sections of a plane a are defined as $a_r = \{x \mid I_r(x, a)\}$ and $a_w = \{x \mid I_w(x, a)\}$, respectively.

Γ2) $J \subset (A \times B) \cup (B \times A)$ is the symmetric incidence between planes and lines. For all $a \in A, b \in B$ it holds $J(a, b) \leftrightarrow \forall x(I(x, b) \rightarrow I_r(x, a) \vee I_w(x, a))$.

Γ3) The intersection of two lines is either empty or a single point.

Γ4) Every line contains infinitely many points. The set of lines is nonempty.

Γ5) For every plane a and every point $c \in a$ there are infinitely many red and infinitely many white lines in a containing c.

Γ6) For every red (resp. white) line b in a plane a there exists exactly one *exceptional point* $c \in a_w \cap b$ (resp. $c \in a_r \cap b$).

Γ7) For every line b and every point $c \in b$ there exist infinitely many planes a with $b \subset a$, such that c is the exceptional point of b in a

Γ8) If $b_1, \dots, b_n$, $n \geq 2$, are pairwise different lines with $b_i \cap b_{i+1} \neq \emptyset$, $1 \leq i \leq n-1$, then $b_1 \cap b_n = \emptyset$, or there exists a point c with $c \in b_i$ for all $i = 1, \dots, n$.

Γ9) Planes are nonempty.

Γ10) The intersection of two planes is either empty or a point or a line.

Γ11) If $a_1, \dots, a_n$, $n \geq 2$, are pairwise distinct planes such that $a_i \cap a_{i+1}$ is a line for $i = 1, \dots, n-1$, then $a_1 \cap a_n = \emptyset$, or $a_1 \cap a_n$ is a point, or $a_1, \dots, a_n$ contain a common line.

Γ12) If a_1, a_2, a_3 are three distinct planes such that $a_i \cap a_j \neq \emptyset$, $1 \leq i, j \leq 3$, then a_1, a_2, a_3 contain a common point.

Γ13) If $a_1, \dots, a_n$, $n \geq 3$, are pairwise distinct planes such that $a_i \cap a_{i+1} \neq \emptyset$, $1 \leq i \leq n-1$, and $a_i \cap a_{i+2} = \emptyset$, $1 \leq i \leq n-2$, then $a_1 \cap a_n = \emptyset$.

To obtain consistent notation we have slightly modified the description of Γ from [HS89] (in [HS89] the symbols A, B, B_r, B_w, C are denoted differently, and incidence relations are not symmetric). The notions of sequences, paths and circles are easily modified to the language L'. It is, however, also allowed to use the direct point-line incidence relations. Therefore, sequences in models of Γ are not necessarily also sequences in the sense of Σ. This can, however, be easily rectified by inserting appropriate lines in the sequences.

In contrast to the pseudospace of Baudisch and Pillay the duality between points and planes is not so apparent from the axioms of Γ. Because of the colors (points are red or white, and planes do not have colors) full duality is not even possible. We will, however, show in the next section that the role of points and planes can be interchanged if colors are omitted.

First we will show the consistency of Γ by constructing a colored version M_0' of the standard model M_0 of Σ from [BP00]. Planes and lines are defined as a free pseudoplane $\omega^{<\omega}$, where the set of planes is $\{\eta \in \omega^{<\omega} \mid \ell(\eta) \text{ is even}\}$ and lines correspond to $\{\eta \in \omega^{<\omega} \mid \ell(\eta) \text{ is odd}\}$. The incidence $J(\eta, \tau)$ holds, if η is a direct predecessor or successor of τ. In analogy to Sect. 2 lines are colored according to $B_r(M_0') = \{b \in B(M_0') \mid b_{\ell(b)} \text{ is even}\}$ and $B_w(M_0') = \{b \in B(M_0') \mid b_{\ell(b)} \text{ is odd}\}$. Hence every plane contains infinitely many red and infinitely many white lines. Planes and lines therefore form a free colored pseudoplane, where the color of planes is neglected.

Now we inductively augment points for the planes and define the relation I. The set of all points is then formed by $\bigcup\{C(a) \mid \ell(a) \text{ even}\}$, such that for every a $(B(a), C(a), I)$ is a connected countable free pseudoplane. Initially, we choose $C(<>)$ as a countable set of points. Colors of $B(<>)$ are already determined by the coloring of (A, B, J) in such a way that $B(<>)$ contains infinitely many red and white lines. On $B(<>) \cup C(<>)$ we define a relation $I_{<>}$ such that $(B(<>), C(<>), I_{<>})$ is a countable connected free colored pseudoplane. In colored pseudoplanes colors were indicated by unary relations $C_r(x)$ and $C_w(x)$, here they are determined via the relations $I_r(x, <>)$ and $I_w(x, <>)$. Colors of $C(<>)$ in planes of length two are chosen in the next step of the construction.

Assume now that $C(a)$ and I_a have already been constructed for all a of length at most $2n$. Let a have length $2n+2$ and let $b = a_{2n+1}$ be the predecessor line of a. Let further C^0 be the set of points of the line b and let C^1 be a countable set of new elements. As in the first step of the induction, the colors of C^0 in planes of length $2n+2$ have not been determined yet. This will be done below, observing axiom $\Gamma 7$).

Now we define I_a on $B(a) \cup C^0 \cup C^1$ such that $(B(a), C^0 \cup C^1, I_a)$ becomes a connected countable free colored pseudoplane. We do not introduce any new points on the line b, i.e., $I_a(b, c)$ holds if and only if $c \in C^0$. Colors of points can be chosen independently in each plane and are indicated by the relations $I_r(x, a)$ and $I_w(x, a)$. Additionally, the exceptional point of b is chosen such that for each $c \in C^0$ there are infinitely many planes a of length $2n+2$ such that $a_{2n+1} = b$ and c is the exceptional point of b on a. This is possible because C^0 is countable and also b contains countably many successor planes a on which the exceptional point can be chosen arbitrarily. Hence $\Gamma 7$) is fulfilled.

Finally, the set of all points M_0' is the union of all sets $C(a)$, and the relation I between points and lines is the union of all I_a. The relations I_r and I_w between points and planes are defined by the respective colorings of the points in the planes.

Let M_0 be the L-reduct of M_0'. It turns out that M_0 is exactly the standard model of Σ constructed in [BP00]. Therefore Σ is valid in M_0'. It remains to show that also Γ is fulfilled in M_0'. The next lemma follows directly from the construction of M_0'.

Lemma 4.1 *M_0' satisfies the axioms $\Gamma 0$) to $\Gamma 2$) and $\Gamma 5$) to $\Gamma 7$).*

The remaining axioms of Γ will be derived from Σ. As M_0' is a model of Σ this implies the validity of Γ in M_0'.

Lemma 4.2 *Every model of Σ satisfies $\Gamma 3$), $\Gamma 4$), and $\Gamma 8$) to $\Gamma 11$).*

Proof. Axiom $\Sigma 1)(b)$ implies $\Gamma 3$), $\Gamma 4$), and $\Gamma 8$). Axiom $\Gamma 9$) is implied by $\Sigma 1)(a)$. The axioms $\Gamma 10$) and $\Sigma 3)(a)$ are identical. Finally, $\Gamma 11$) follows from $\Sigma 1)(a)$ and $\Sigma 3)(a)$. □

To derive $\Gamma 12$) and $\Gamma 13$) from Σ requires some extra arguments.

Lemma 4.3 *Every model of Σ satisfies $\Gamma 12$).*

Proof. Let a_1, a_2, a_3 be distinct planes and let c_1, c_2, c_3 be points such that $c_1 \in a_2 \cap a_3$, $c_2 \in a_1 \cap a_3$, $c_3 \in a_1 \cap a_2$. We have to show the existence of a point $c \in a_1 \cap a_2 \cap a_3$. If $c_1 \in a_1$, then $c = c_1$ is such a point. Likewise, if $c_2 \in a_2$ or $c_3 \in a_3$. Assume now that

$$c_i \notin a_i \text{ for } i = 1, \ldots, 3. \tag{1}$$

We will derive a contradiction. By assumption c_1, c_2, and c_3 are pairwise distinct. Hence there exists a circle

$$(a_1, b_3, c_3, b_3', a_2, b_1, c_1, b_1', a_3, b_2, c_2, b_2', a_1)$$

with pairwise distinct $b_1, b_2, b_3, b_1', b_2', b_3'$. Choosing such lines is possible by (1) and because $(B(a_i), C(a_i), I)$ is a free pseudoplane. By $\Sigma 4)(a)$ there exists a BC-path X between b_3 and b_2', containing only lines from a_1 and points from $\{c_1, c_2, c_3\}$. The point c_1 cannot occur in X because $c_1 \notin a_1$. Hence we have either

(a)	$X = (b_3, c_3, b_2')$	or
(b)	$X = (b_3, c_2, b_2')$	or
(c)	$X = (b_3, c_3, b', c_2, b_2')$	with $b' \subset a_1$.

In every case there exists a line $b_1'' \subset a_1$ with $c_3 \in b_1''$ and $c_2 \in b_1''$, namely in (a) $b_1'' = b_2'$, in (b) $b_1'' = b_3$ and in (c) $b_1'' = b'$. Analogously, using $c_2 \notin a_2$ and $c_3 \notin a_3$ we get lines $b_2'' \in a_2$ and $b_3'' \in a_3$ with $c_1, c_3 \in b_2''$ and $c_1, c_2 \in b_3''$. By (1) the lines b_1'', b_2'' and b_3'' are pairwise distinct. Hence there exists a circle of lines and points

$$(b_1'', c_3, b_2'', c_1, b_3'', c_2, b_1'')\ ,$$

contradicting $\Sigma 1)(b)$. □

Lemma 4.4 *Every model of* Σ *satisfies* $\Gamma 13)$.

Proof. Let $a_1, \ldots, a_n$, $n \geq 3$, be distinct planes with $c_i \in a_i \cap a_{i+1}$, $1 \leq i < n$, and $a_i \cap a_{i+2} = \emptyset$, $1 \leq i < n-1$. We have to prove $a_1 \cap a_n = \emptyset$. We will show this by induction on n. The base case $n = 3$ is clear. Let $n > 3$ and assume that $\Gamma 13)$ is valid for all $3 \leq k < n$. By hypothesis we have

$$a_i \cap a_j = \emptyset \text{ for } 1 \leq i < j \leq n \text{ with } i+1 \neq j \text{ and } (i,j) \neq (1,n). \qquad (2)$$

Assume now, that there exists a point $c_n \in a_1 \cap a_n$. We will construct a contradiction, similarly as in the previous lemma. By (2) and the assumption there exists a circle

$$(a_1, b, c_1, b', a_2, \ldots, a_n, b'', c_n, b''', a_1)\ .$$

Applying $\Sigma 4)(a)$ yields a path X between b and b''', containing only lines from a_1 and points from $\{c_1, \ldots, c_n\}$. By (2) $c_2, \ldots, c_{n-1}$ cannot appear in X. Hence we have $X = (b, c_1, b''')$, or $X = (b, c_n, b''')$, or $X = (b, c_1, b_1, c_n, b''')$ with $b_1 \subset a_1$. In each case there exists a line b_1, containing the points c_1 and c_n. Analogously, (2) yields lines $b_2, \ldots, b_n$ with $c_{i-1} \in b_i$ and $c_i \in b_i$, $2 \leq i \leq n$. By (2) all lines $b_1, \ldots, b_n$ are distinct. Hence there is a circle

$$(b_1, c_1, b_2, c_2, \ldots, b_n, c_n, b_1)\ ,$$

contradicting axiom $\Sigma 1)(b)$. □

Corollary 4.5 M_0' *is a model of* Γ, *and hence* Γ *is consistent.*

5 The Relationship Between the Two Pseudospaces

In this section we analyse the relationship between the axioms of Σ and Γ. Already in the last section we have shown that most axioms from Γ (except those concerning the colors) are derivable from Σ. Now we will prove that the axioms of Γ also imply all axioms from Σ. But first we will make two remarks on the system Σ itself.

Lemma 5.1 *Every model of* $\Sigma 0), \Sigma 1)$ *and* $\Sigma 2)(a)$ *fulfills* $\Sigma 2)(b)$.

Proof. Let c be a point. We have to show that $(A(c), B(c), J)$ is a free pseudoplane, i.e., we have to check the axioms $\Delta 1)$ to $\Delta 4)$. Axioms $\Delta 1)$ and $\Delta 2)$ follow immediately from $\Sigma 0)$.

For $\Delta 3)$ let $a \in A(c)$. By $\Sigma 2)(a)$ $(B(a), C(a), I)$ is a free pseudoplane. Because $c \in C(a)$ there exist infinitely many lines in a that contain c. Therefore every plane in $(A(c), B(c), J)$ contains infinitely many lines. That every line b lies in infinitely many planes follows from $\Sigma 1)(a)$, because every plane $a \supset b$ also contains c. Finally, $(A(c), B(c), J)$ does not contain circles as this is already true for (A, B, J) by $\Sigma 1)(a)$. Hence also $\Delta 4)$ is fulfilled. □

Lemma 5.2 *Every model of* $\Sigma 0)$, $\Sigma 1)$ *and* $\Sigma 3)(a)$ *satisfies* $\Sigma 3)(b)$.

Proof. Let c and c' be two distinct points. If there is none or exactly one plane containing c and c', then $\Sigma 3)(b)$ is already fulfilled for c and c'.

Assume therefore that a and a' are two distinct planes that both contain c and c'. Then $\{c, c'\} \subseteq a \cap a'$, and by $\Sigma 3)(a)$ there exists a line b such that $c, c' \in b$ and $a \cap a' = b$. By $\Sigma 1)(b)$ this line b is uniquely determined by c and c'. Hence the planes containing c and c' are exactly the planes that contain b. □

For the axioms $\Gamma 5)$ and $\Gamma 7)$ we will now consider the weaker assertions $\Gamma 5')$ and $\Gamma 7')$.

$\Gamma 5')$ For every plane a and every point $c \in a$ there exist infinitely many lines in a that contain c.

$\Gamma 7')$ Every line lies in infinitely many planes.

$\Gamma 0')$ and $\Gamma 2')$ are obtained from $\Gamma 0)$ and $\Gamma 2)$ by omitting the parts that refer to the color relations B_r, B_w and I_r, I_w. The system Γ' in the language L consists of the axioms $\Gamma 0')$, $\Gamma 2')$, $\Gamma 3)$, $\Gamma 4)$, $\Gamma 5')$, $\Gamma 7')$, and $\Gamma 8)$ to $\Gamma 13)$. Apparently we have:

Proposition 5.3 *Every model of* Γ *is a model of* Γ'.

We now aim to show the equivalence of Σ and Γ'. For this we will first derive $\Sigma 4)(a)$ from Γ', which requires the following lemma.

Lemma 5.4 *Let* $M \models \Gamma'$ *and let* $X = (a, b, c_0, a_1, c_1, \ldots, c_{n-1}, a_n, c_n, b', a)$ *be a circle in* M *consisting of planes* $a = a_0, a_1, \ldots, a_n$, *lines* b, b', *and points* $c_0, \ldots, c_n$. *Then there exists a BC-path* $Y = (b, c'_0, b'_1, c'_1, \ldots, c'_{m-1}, b'_m, c'_m, b')$ *such that* $\{c'_0, \ldots, c'_m\} \subseteq \{c_0, \ldots, c_n\}$ *and* $c'_i \in a$, $0 \leq i \leq m$. *Additionally, we have* $b'_i \subset a$, $1 \leq i \leq m$.

Proof. The last sentence follows from the first part of the lemma. Namely, if $a'_i \neq a$ is a plane with $b'_i \subset a'_i$, then $c'_{i-1}, c'_i \in a'_i \cap a$, and hence $b'_i = a'_i \cap a$.

The first part of the lemma is shown by induction on n. Because for $n = 2$ we use Γ12), and we can only use Γ13) for $n \geq 3$, we have to include also $n = 2$ in the base case of the induction.

Base case. For $n = 0$ we have $X = (a, b, c_0, b', a)$, and the claim is true.

For $n = 1$ we have $X = (a, b, c_0, a_1, c_1, b', a)$, i.e., $c_0, c_1 \in a \cap a_1$. Then there exists a line $b'' = a \cap a_1$, and hence there is the sequence (b, c_0, b'', c_1, b'). If $b = b''$ or $b'' = b'$, then the sequence can be shortened. In the following we will not explicitly mention, if a sequence can be shortened in such a way.

For $n = 2$ we have $X = (a, b, c_0, a_1, c_1, a_2, c_2, b', a)$. By Γ12) there exists a point $c \in a \cap a_1 \cap a_2$. We will distinguish four cases.

Case 1. $c = c_0$, i.e., in particular $c \neq c_2$. Then there exists $b'' = a \cap a_2$ such that $c_0, c_2 \in b''$. Therefore the desired path Y is obtained from the sequence (b, c_0, b'', c_2, b').

Case 2. $c = c_1$, hence $c \neq c_0$ and $c \neq c_2$. Then there exist lines $b'' = a \cap a_1$ and $b''' = a \cap a_2$ such that $c_0, c_1 \in b''$ and $c_1, c_2 \in b'''$. Therefore we have the sequence $(b, c_0, b'', c_1, b''', c_2, b')$.

Case 3. $c = c_2$. Like case 1.

Case 4. $c \neq c_0$, $c \neq c_1$, and $c \neq c_2$. Then we have lines $b_0 = a \cap a_1$, $b_1 = a_1 \cap a_2$, and $b_2 = a_2 \cap a$, i.e., there exists the sequence $(a, b_0, a_1, b_1, a_2, b_2, a)$. Because a, a_1, a_2 are pairwise distinct they contain a common line b'' by Γ11), which is identical with b_0, b_1 and b_2 by Γ10), i.e., $b'' = b_0 = b_1 = b_2$. Therefore we get the sequence (b, c_0, b_0, c_2, b').

Induction step. Let the claim be true for $k < n$, $n \geq 3$. If there exists an index i, $1 \leq i \leq n-1$ such that $c_i \in a$, then we choose $b'' \subset a$ with $c_i \in b''$. Hence we have the paths $(a, b, c_0, \ldots, a_i, c_i, b'', a)$ and $(a, b'', c_i, a_{i+1}, \ldots, a_n, c_n, b', a)$. By induction hypothesis there exist BC-paths connecting b and b'' as well as b'' and b', that only use points from $\{c_0, \ldots, c_n\}$ lying in a. We obtain the desired BC-path between b and b' by concatenation. We can therefore make the following

Assumption 1 $c_i \notin a$ *for* $1 \leq i \leq n-1$.

If there exist i and j such that $0 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j, j-1$ and $c_i \in a_j$, then we can shorten the path X to $X' = (a, b, c_0, \ldots, a_i, c_i, a_j, c_j, \ldots, b', a)$ if $i < j-1$, and to $X' = (a, b, c_0, \ldots, c_{j-1}, a_j, c_i, a_{i+1}, \ldots, b', a)$ if $i > j$. In this case the induction hypothesis for X' yields the claim. In addition to Assumption 1 we therefore make

Assumption 2 $c_i \notin a_j$ *for* $0 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j, j-1$.

Finally, if c_0 and c_n are on a common line b'', then we directly get the path Y from (b, c_0, b'', c_n, b'). We therefore also assume

Assumption 3 c_0 *and* c_n *do not lie on a common line.*

From Assumptions 1 to 3 we will derive a contradiction, hence for any given X at least one of these assumptions does not hold, and thus the claim is proved. By Γ13) there exists some j, $0 \leq j \leq n-2$ such that $a_j \cap a_{j+2} \neq \emptyset$. Let $c \in a_j \cap a_{j+2}$. For this situation we will prove the following claim.

Claim 1 *If there exists a point $c \in a_j \cap a_{j+2}$ with $0 \leq j \leq n-2$, then there exists a BC-path Y' which connects b and b' and does not use any points except c_0, c_n, and c. Further, c appears in Y', and we have $c \neq c_0$, $c \neq c_n$, and $c \in a$.*

Proof of Claim 1. To prove the first sentence we will distinguish two cases.

Case 1. $j = 0$, i.e., $c \in a \cap a_2$. Then there exists $b'' \subset a$ such that $c \in b''$. Hence we get circles $(a, b, c_0, a_1, c_1, a_2, c, b'', a)$ and $(a, b'', c, a_2, \ldots, a_n, c_n, b', a)$. Applying the induction hypothesis twice yields BC-paths between b and b'' as well as between b'' and b'. By Assumption 1 these paths only contain points from $\{c_0, c_n, c\}$. By concatenation we get a BC-path connecting b and b'.

Case 2. $1 \leq j \leq n-2$. Then we have a circle $(a, b, \ldots, a_j, c, a_{j+2}, \ldots, b', a)$, and by induction hypothesis and Assumption 1 we get a BC-path that contains only the points c_0, c_n, c.

Thus the path Y' only uses the points c_0, c_n, and c. The point c is included in Y' because all choices for Y' omitting c, i.e., (b, c_0, b'), (b, c_n, b'), and (b, c_0, b'', c_n, b') contradict Assumption 3. Thus the only possible configurations for Y' are (b, c, b'), (b, c, b'', c_n, b'), and $(b, c_0, b'', c, b''', c_n, b')$, in which case $c \neq c_0$ and $c \neq c_n$ follow by Assumption 3 and because Y' is a path. Hence c appears in Y', and therefore we get $c \in a$ by induction hypothesis. □

By Γ13) there exists a point $c \in a_j \cap a_{j+2}$, $0 \leq j \leq n-2$. Applying Claim 1 to this point we obtain a BC-path between b and b', using the point c and possibly also c_0 and c_n. We will call this path Y_1. In the next claim we prove that c is contained in all planes a_i.

Claim 2 *For all $1 \leq i \leq n$ we have $c \in a_i$.*

Proof of Claim 2. We will prove inductively the following claim: if $c \in a_i$ and $c \in a_j$, $0 \leq i < j \leq n+1$ (with $a_0 = a_{n+1} = a$), then $c \in a_k$ for all $i \leq k \leq j$. The proof proceeds by induction on $l = j - i$.

Base case. For $l = 1$ there is nothing to show. Let $l = 2$, i.e., $c \in a_i \cap a_{i+2}$. By Γ12) we have $a_i \cap a_{i+1} \cap a_{i+2} \neq \emptyset$. Let $c' \in a_i \cap a_{i+1} \cap a_{i+2}$. Claim 1 for c' yields a BC-path Y_2 between b and b', that contains c' and possibly also c_0 and c_n. As (B, C, I) is a free pseudoplane, BC-paths are unique, and therefore $Y_1 = Y_2$ and in particular $c = c'$. Hence $c \in a_{i+1}$.

Induction step. Let $l \geq 3$ and let the claim be true for all $k < l$. Then we have the situation $(a_i, c_i, \ldots, c_{i+l-1}, a_{i+l}, c, a_i)$. By Γ13) there exists an index m such that $i \leq m \leq i + l - 2$ and $a_m \cap a_{m+2} \neq \emptyset$. Let $c' \in a_m \cap a_{m+2}$. As before, applying Claim 1 to c' we get a BC-path Y_2 between b and b', using only the points c_0, c_n and c'. Then we have $Y_1 = Y_2$ and hence $c = c'$. Therefore $c \in a_i \cap a_{m+1} \cap a_{i+l}$ and the induction hypothesis yields $c \in a_{i+k}$, $0 \leq k \leq l$. □

By Claim 1 we have $c \in a$ and by Assumption 1 $c \neq c_i$, $1 \leq i \leq n-1$. Together with Claim 1 we get $c \neq c_i$, $0 \leq i \leq n$. By Claim 2 this means

$c, c_i \in a_i \cap a_{i+1}$, $0 \leq i \leq n$. Thus by Γ10) there exist lines $b_i = a_i \cap a_{i+1}$ such that $c, c_i \in b_i$, $0 \leq i \leq n$. The lines $b_0, \ldots, b_{n-1}$ are pairwise distinct, because if $b_i = b_j$, $0 \leq i < j \leq n-1$, then $c_i \in b_i = b_j = a_j \cap a_{j+1}$ in contradiction to Assumption 2. By the same argument the lines $b_1, \ldots, b_n$ are pairwise distinct. Additionally, Assumption 3 yields $b_0 \neq b_n$, hence all of $b_0, \ldots, b_n$ are pairwise distinct. Therefore we get an AB-circle $(a, b_0, a_1, b_1, \ldots, b_{n-1}, a_n, b_n, a)$ in contradiction to Γ11). Hence Assumptions 1 to 3 cannot hold simultaneously, and the proof is complete. □

This enables us to prove the validity of Σ4)(a) in Γ.

Theorem 5.5 *Every model of* Γ' *satisfies* Σ4)(a).

Proof. Let $X = (a, b, \ldots, b', a)$ be an ABC-circle. We have to construct a BC-path Y connecting b and b' and consisting only of points from X which are in a. To apply the previous lemma we transform X to a circle X' that contains no lines except b and b'. To achieve this we apply the following steps a) to c) to the inner part $b, \ldots, b'$ of X:

a) Every sequence of the form a_1, b_1, c_1 is replaced by a_1, c_1. Similarly, every sequence c_1, b_1, a_1 is shortened to c_1, a_1.

b) Every sequence c_1, b_1, c_2 is substituted by c_1, a_1, c_2, where the plane a_1 contains the line b_1 and does not occur in X.

c) Finally, every sequence of the form a_1, b_1, a_2 is changed to a_1, c_1, a_2 with an arbitrary point c_1 from b_1 that does not occur in X.

After these steps have been performed on X we apply the following rule:

d) If the circle X obtained after the steps a) to c) starts with a, b, a_1, then we choose some point c_0 from b, not contained in X, and replace a, b, a_1 by a, b, c_0, a_1. Similarly, if X ends with a_n, b', a, then we insert a new point $c_n \in b'$, obtaining a_n, c_n, b', a.

The circle X' thus obtained has the form $X' = (a, b, c_0, a_1, c_1, \ldots, a_n, c_n, b', a)$ and contains only planes, the lines b and b', and all points from X as well as new points inserted by the rules c) and d). Applying Lemma 5.4 to X' yields a BC-path Y between b and b' with points from X' and lines from a. The new points introduced by the rules c) and d) can be chosen arbitrarily from infinitely many possibilities. Therefore, as BC-paths are unique, these new points cannot appear in Y. Hence the path Y is in accordance with the requirements from axiom Σ4)(a). □

It remains to derive axiom Σ4)(b) from Γ. This requires a similar result as Lemma 5.4, but with a considerably simpler proof.

Lemma 5.6 *In a model of* Γ' *let* $X = (c, b, a_0, c_0, \ldots, c_{n-1}, a_n, b', c)$ *be a circle consisting of planes* $a_0, \ldots, a_n$, *lines* b, b', *and points* $c_0, \ldots, c_{n-1}$. *Then there exists an* AB*-path* $Y = (b, a'_0, b'_0, \ldots, b'_{m-1}, a'_m, b')$ *with* $\{a'_0, \ldots, a'_m\} \subseteq \{a_0, \ldots, a_n\}$ *and* $c \in a'_i$, $0 \leq i \leq m$. *Additionally, we have* $c \in b'_i$, $0 \leq i \leq m-1$.

Proof. The last assertion $c \in b'_i$ follows from $c \in a'_i \cap a'_{i+1} = b'_i$. We will show the first part of the claim by induction on n.

Base case. For $n = 0$ we have $X = (c, b, a_0, b', c)$, and the claim holds.

For $n = 1$ we have $X = (c, b, a_0, c_0, a_1, b', c)$. Because $c, c_0 \in a_0 \cap a_1$ there is a line $b_0 = a_0 \cap a_1$, and hence we get the sequence (b, a_0, b_0, a_1, b').

For $n = 2$ we have $X = (c, b, a_0, c_0, a_1, c_1, a_2, b', c)$. By Γ12) there exists a point $c' \in a_0 \cap a_1 \cap a_2$. We will distinguish four cases.

Case 1. $c' = c_0$. Then $c_0 \in a_2$, and there exists the circle $(c, b, a_0, c_0, a_2, b', c)$. We can therefore continue as in the case $n = 1$.

Case 2. The case $c' = c_1$ is analogous to case 1.

Case 3. $c' = c$, and therefore in particular $c' \neq c_0$ and $c' \neq c_1$. Then there exist lines $b_0 = a_0 \cap a_1$ and $b_1 = a_1 \cap a_2$, yielding the path $(b, a_0, b_0, a_1, b_1, a_2, b')$.

Case 4. $c' \neq c$, $c' \neq c_0$, and $c' \neq c_1$. Then there exist lines $b_0 = a_0 \cap a_1$, $b_1 = a_1 \cap a_2$, and $b_2 = a_2 \cap a_0$, i.e., we have the sequence $(a_0, b_0, a_1, b_1, a_2, b_2, a_0)$. From this we infer $b_0 = b_1 = b_2$ (cf. the resp. part of the proof of Lemma 5.4), and thus we obtain the path (b, a_0, b_0, a_2, b').

Induction step. Let the claim be true for $k < n$, $n \geq 3$. By Γ13) there exists an index i, $0 \leq i \leq n - 2$ such that $a_i \cap a_{i+2} \neq \emptyset$. Let $c' \in a_i \cap a_{i+2}$. Applying the induction hypothesis to $(c, b, a_0, \ldots, c_{i-1}, a_i, c', a_{i+2}, c_{i+2}, \ldots, a_n, b', c)$ yields the desired AB-path Y. □

Theorem 5.7 *Every model of* Γ' *satisfies* $\Sigma 4)(b)$.

Proof. Let $X = (c, b, \ldots, b', c)$ be an ABC-circle. We search for an AB-path between b and b' with planes from X that contain c. As the proof of Theorem 5.5 we transform X to a circle X' that does not contain any lines except b and b'. This is achieved by first applying the rules a) to c) from the proof of Theorem 5.5 to the inner part $b, \ldots, b'$ of X. Rule d) is replaced by:

- d') If the path X obtained by rules a) to c) starts with c, b, c_0, then we choose a plane a_0 with $b \subset a_0$ and replace c, b, c_0 by c, b, a_0, c_0. Similarly, if the path ends with c_{n-1}, b', c, then we insert a new plane $a_n \supset b'$.

The path X' obtained from X by the rules a) to c) and d') is now of the form $X' = (c, b, a_0, c_0, \ldots, a_{n-1}, c_{n-1}, a_n, b', c)$. Applying Lemma 5.6 to X' yields an AB-path Y between b and b' with planes from X' and lines containing c. The planes inserted by rules b) and d') cannot appear in Y, because AB-paths are unique, and for the new planes from b) and d') there exist infinitely many different choices. Therefore Y meets the requirements of $\Sigma 4)(b)$. □

These preparations enable us to characterize the relationship between Σ and Γ as follows:

Theorem 5.8 Σ *and* Γ' *are equivalent.*

Proof. In the last section we have already shown that Σ implies the axioms Γ3),Γ4), and Γ8) to Γ13). The remaining axioms of Γ' are also easily seen to be valid in models of Σ, and therefore $\Sigma \models \Gamma'$.

Concerning the converse $\Gamma' \models \Sigma$, axiom $\Sigma 0)$ follows from $\Gamma 0')$ and $\Gamma 2')$. To derive $\Sigma 1)$ and $\Sigma 2)(a)$ from Γ' we have to check the axioms Δ for the respective pseudoplanes. Axioms $\Delta 1)$ and $\Delta 2)$ defining the incidence relations are apparently fulfilled. $\Delta 3)$ is easily checked to follow from $\Gamma 4)$, $\Gamma 5')$, and $\Gamma 7')$. For $\Delta 4)$ we have to verify the absence of circles. AB-circles do not exist by $\Gamma 10)$ and $\Gamma 11)$. By $\Gamma 3)$ and $\Gamma 8)$ there are also no BC-circles. Clearly, then there are also no circles in the pseudoplanes mentioned in $\Sigma 2)(a)$.

$\Sigma 3)(a)$ is equivalent to $\Gamma 10)$. By Lemmas 5.1 and 5.2 the axioms $\Sigma 2)(b)$ and $\Sigma 3)(b)$ hold in Γ'. Finally, $\Sigma 4)$ was proved in Theorems 5.5 and 5.7. □

Corollary 5.9 *Σ is the L-reduct of Γ, i.e., if M is an L'-structure such that $M \models \Gamma$, then $M|L$ is a model of Σ.*

This corollary also clarifies the duality between points and lines in models of Γ, namely, if the colors are removed, then points and planes can be interchanged. In fact, this duality is a very natural concept, that does not even have to be required axiomatically. This is the content of the next corollary.

Corollary 5.10 *Every model of $\Sigma 0), \Sigma 1), \Sigma 2)(a), \Sigma 3)(a)$, and $\Sigma 4)(a)$ fulfills all axioms of Σ.*

Proof. Let M be an L-structure satisfying $\Sigma 0), \Sigma 1), \Sigma 2)(a), \Sigma 3)(a)$, and $\Sigma 4)(a)$. By Lemmas 5.1 and 5.2 M also satisfies $\Sigma 2)(b)$ and $\Sigma 3)(b)$. In the proof of $\Sigma \models \Gamma'$ we only used the above mentioned axioms of Σ. In particular, the proofs of Lemmas 4.3 and 4.4 do not involve $\Sigma 4)(b)$. Therefore $M \models \Gamma'$, and with Theorem 5.8 we get $M \models \Sigma$. □

6 On the Non-Equationality of Γ

Baudisch and Pillay proved in [BP00] that the pseudospace Σ is a complete and ω-stable theory. Once we know that Γ is a reduct of Σ, the same line of arguments can be also used to show the completeness and ω-stability of Γ. This involves in particular exploring the fine structure of sufficiently saturated models of Γ and a detailed type analysis together with the computation of Morley ranks. In comparison to [BP00], however, the details are somewhat more tedious due to the richer language of Γ. We will omit this altogether and proceed to explain the original purpose of Γ as an example of a stable and non-equational theory.

Computing Morley ranks in Γ it turns out that, as in Σ, the Morley rank of a plane a is ω. However, in contrast to Σ, where we have $MD(a) = 1$, the Morley degree of a increases to 2 in Γ, owing to the fact that a splits into a white and a red section. For these we get $MR(a_r) = MR(a_w) = \omega$ and $MD(a_r) = MD(a_w) = 1$. Building on this analysis the next result from [HS89] is the key lemma for showing the non-equationality of Γ. In fact, it is the only place in the whole argument where equations come into play.

Lemma 6.1 (Hrushovski, Srour [HS89]) *Let $\varphi(x, \bar{y})$ be an equation and $D = \varphi(x, \bar{d})$ be an instance of this equation. Then for every line b and every plane a the following holds:*

1. *Let b be almost in D, i.e., all points of b except finitely many are in D. Then already all points of b are in D.*

2. *If $MR(a_r \setminus D) < \omega$, then $a \subseteq D$.*

Proof. For the first item let us assume that there exists a point $c \in b \setminus D$, and let c' be an arbitrary point from b. In the type analysis of Γ it turns out that points are indiscernible over lines, i.e., there exists an automorphism f mapping c to c' and fixing b. We will denote $f(D)$ by $D_{c'}$. Because $c \notin D$ we also have $c' \notin D_{c'}$. As b is almost in D and is fixed by f, the line b is also almost in $D_{c'}$. Varying the point c' we get $\bigcap_{c' \in b} D_{c'} = \emptyset$, because $c' \notin D_{c'}$. The sets $D_{c'}$ are all instances of the equation φ, hence there exist points $c_1, \ldots, c_n$ from b such that

$$\bigcap_{i=1}^{n} D_{c_i} = \bigcap_{c' \in b} D_{c'} = \emptyset \ .$$

But by assumption b is almost in D_{c_i} for $1 \leq i \leq n$ and therefore also almost in $\bigcap_{i=1}^{n} D_{c_i}$, which gives a contradiction.

For part 2 we first prove $a_r \subseteq D$ by a similar argument as in part 1. Assume that there exists a point $c \in a_r \setminus D$, and let $c' \in a_r$ be arbitrary. As before there exists an automorphism f such that $f(c) = c'$ and $f(a) = a$. Let again $D_{c'}$ denote $f(D)$. By $c \notin D$ we get $c' \notin D_{c'}$. As f also fixes the red section a_r we have $f(a_r \setminus D) = a_r \setminus D_{c'}$. Morley ranks are preserved by automorphisms, hence $MR(a_r \setminus D_{c'})$ is finite. As $D_{c'}$ are instances of the equation φ, there exist points $c_1, \ldots, c_n$ such that $\bigcap_{i=1}^{n} D_{c_i} = \bigcap_{c' \in a_r} D_{c'} = \emptyset$. Therefore $\bigcup_{i=1}^{n} a_r \setminus D_{c_i} = a_r$ and $MR(a_r \setminus D_{c_i}) < \omega$, contradicting $MR(a_r) = \omega$. This shows $a_r \subseteq D$.

It remains to show $a_w \subseteq D$. For this let $c \in a_w$. There exists a red line b in a that contains c, i.e., c is the exceptional point of b in a. Then b is contained almost in a_r, hence also almost in D. By part 1 we conclude that the whole line b lies in D, hence in particular $c \in D$. □

This lemma enables us to give the full proof of the main theorem of [HS89] stating the non-equationality of the pseudospace Γ.

Theorem 6.2 (Hrushovski, Srour [HS89]) Γ *is not equational. More precisely, if a is a plane, then the formula $I_r(x, a)$, defining the red section a_r of a, is not equivalent to a Boolean combination of equations.*

Proof. Assume that a_r can be described by a Boolean combination of equations, and let

$$a_r = \bigvee_{i=1}^{n} \left(\bigwedge_{j=1}^{n_i} \psi_{ij} \wedge \bigwedge_{j=1}^{n'_i} \neg \psi'_{ij} \right) ,$$

where ψ_{ij} and ψ'_{ij} are instances of equations. Finite conjunctions and finite disjunctions of equations are again equations (cf. [Jun00]). Using the abbreviations $\psi_i = \bigwedge_{j=1}^{n_i} \psi_{ij}$ and $\psi'_i = \bigvee_{j=1}^{n'_i} \psi'_{ij}$ we can therefore write a_r as

$$a_r = \bigvee_{i=1}^{n} \psi_i \wedge \neg\psi'_i$$

with equations ψ_i and ψ'_i.

Because $MR(a_r) = \omega$, there exists an index j, $1 \leq j \leq n$, such that $MR(\psi_j \wedge \neg\psi'_j) = \omega$. Let $Y = \psi_j \wedge \neg\psi'_j$. From $MD(a_r) = 1$ and $MR(a_r) = MR(Y) = \omega$ we conclude $MR(a_r \setminus Y) < \omega$.

Because $a_r \setminus \psi_j \subseteq a_r \setminus Y$ we get $MR(a_r \setminus \psi_j) \leq MR(a_r \setminus Y)$, hence in particular $MR(a_r \setminus \psi_j)$ is finite. Part 2 of Lemma 6.1 then yields $a \subseteq \psi_j$. As $Y \subseteq a_r$ this implies $a_w \subseteq \psi'_j$. As in the proof of part 2 of Lemma 6.1 this extends to $a \subseteq \psi'_j$. Namely, if $c \in a_r$, then there exists a white line b in a such that c is the exceptional point of b in a. As b is almost in ψ'_j we get by part 1 of Lemma 6.1 $b \subseteq \psi'_j$ and hence $c \in \psi'_j$. Now we have $Y \subseteq a_r$ and $a \subseteq \psi'_j$ which implies $Y = \emptyset$. But this means $MR(a_r) = MR(a_r \setminus Y) < \omega$ in contradiction to $MR(a_r) = \omega$. □

The free pseudospace Γ is so far the only known example of a stable and non-equational theory. Already Hrushovski and Srour remark in [HS89] that, although Γ is not equational, it is almost equational, a weakening of equationality where the forking relation is controlled by equations (cf. [JK02]). It remains as an open problem to construct a theory that is simple (or even stable) but not almost equational.

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