The Complexity of Graph Isomorphism for Colored Graphs with Color Classes of Size 2 and 3

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Abstract. We prove that the graph isomorphism problem restricted to colored graphs with color multiplicities 2 and 3 is complete for symmetric logarithmic space SL under many-one reductions. This result improves the existing upper bounds for the problem.
We also show that the graph automorphism problem for colored graphs with color classes of size 2 is equivalent to deciding whether an undirected graph has more than a single connected component and we prove that for color classes of size 3 the graph automorphism problem is contained in SL.

1 Introduction

The graph isomorphism problem GI consists in deciding whether there is a bijection between the nodes of two given graphs, G and H, preserving the edge relation. GI is one of the most intensively studied problems in Theoretical Computer Science. Besides its many applications, the main source of interest in GI has been the evidence that this problem is probably neither in P nor NP-complete. Other sources of interest include the sophistication of the tools developed to attack the problem (for example [3, 14]), and the connections between GI and structural complexity (see [10]).

Understandably, many GI restrictions have been considered. For example, P upper bounds are known in the cases of planar graphs [7] or graphs of bounded valence [14]. In some cases, like trees [12, 4] or graphs with colored vertices and bounded color classes [15], even NC algorithms for isomorphism exist (NC^k is defined to be the class of problems solvable by a uniform family of bounded indegree Boolean circuits with $O(\log^k n)$ depth and polynomial size, and $\mathrm{NC} = \bigcup_{k\geq 0} \mathrm{NC}^k$).

However, until recently none of these GI restrictions were known to be complete for a natural complexity class, and it seemed that these problems lack the structure needed for a hardness result. Several results in the last years have changed this situation. In [9] it was shown that the isomorphism problem for trees is hard for NC^1 and for L (logarithmic space) depending on the encoding of the input. Moreover in [17] it was proved that isomorphism for general graphs is hard for NL (nondeterministic logarithmic space) and for every logarithmic space class based on counting. Regarding completeness, two versions of tree isomorphism are respectively complete for NC^1 and L [12,4,9] and until now this was the only GI restriction to a special graph family known to be complete for a complexity class.

We continue in this paper with this line of research concentrating on the family of graphs with colored nodes and with a small number of nodes with the same color. We will denote by b-GI the isomorphism problem for colored graphs with color multiplicities bounded by b. Isomorphisms between colored graphs are color preserving and this reduces the number of possible isomorphisms. But observe that even for the case $b = 2$ the number of color-preserving mappings between n-vertex graphs could be as large as $2^{\frac{n}{2}}$. This problem has a fairly long tradition in the area of algorithms. Babai gave in [2] a random polynomial-time algorithm for testing graph isomorphism in the case that the color multiplicities are bounded by a constant b. The algorithm was improved to be deterministic in [6] and, by using sophisticated algebraic tools, Luks proved in [15] that b-GI lies in fact in NC.

We focus in this paper in the family of colored graphs with color multiplicities bounded by the constants 2 and 3 proving that 2-GI and 3-GI are many-one complete for symmetric logarithmic space SL under logarithmic space reductions. The complexity class SL introduced in [11] has different characterizations, but the easiest way to define it is the following: SL is the class of problems that are logarithmic space reducible to the reachability problem for undirected graphs, UGAP. Our results improve on the one hand the upper bound for the problem given by Luks from NC to SL (SL is included in NL and in NC^2). On the other hand the results provide new natural examples for complete problems in SL showing that reachability questions can be expressed in a natural way as isomorphisms in a class of graphs.

Closely related to graph isomorphism is the graph automorphism problem GA that consists in deciding whether a given graph has a non-trivial automorphism, or in other words, whether there is a permutation of the nodes, different from the identity, preserving the adjacency relation. The relationship between GA and GI is not completely clear. It is known that GA is many-one reducible to GI [13] but a reduction in the other direction is not known and GA seems to be an easier problem. We show that this situation also occurs when we restrict the problem to colored graphs with color classes of size 2 and 3. We prove that 2-GA and 3-GA belong also to SL and moreover 2-GA is equivalent to the problem $\overline{\mathrm{UCC}}$ of deciding whether a given graph has more than one connected component. $\overline{\mathrm{UCC}}$ belongs to SL but it seems to be easier than UGAP and it is not known to be complete for SL [1].

For simplicity we have proven our hardness and equivalence results for logarithmic space many-one reducibility, but in fact they hold for stronger reducibilities, like for example DLOGTIME uniform NC^1 many-one reducibility. All the graphs considered in this paper are undirected graphs.

2 Upper bound for 2-GA

We show in this section that 2-GA is reducible to the problem $\overline{\mathrm{UCC}}$ of deciding whether a given graph has more than a single connected component. For a graph G we denote by $\mathrm{Aut}(G)$ the set of automorphisms in G. The next lemma shows that we can restrict ourselves to graphs with at most two edges between any two colors. The proof of this result is straightforward.

Lemma 1. *Let $G = (V, E)$ be a graph with colored vertices, and let C_i and C_j two color classes in G. Then $\mathrm{Aut}(G) = \mathrm{Aut}(G')$, where $G' = (V, E')$ is a copy of G but with the dual set of edges between vertices of C_i and C_j (for every pair (u, v) with $u \in C_i$ and $v \in C_j$, it holds $(u, v) \in E \Leftrightarrow (u, v) \notin E'$).*

Theorem 1. *The Graph Automorphism problem restricted to graphs with color classes of size at most 2 can be reduced to $\overline{\mathrm{UCC}}$.*

Proof. Let G be a graph with color classes of size at most 2. By the above lemma we can assume that for every pair of colors, there are at most two edges connecting the nodes of these colors. Also, w.l.o.g. we can assume that there is no edge between the nodes of the same color in G (these edges do not change the automorphism group). The possible connections that have to be considered between the nodes of two different colors are shown in Figure 1.

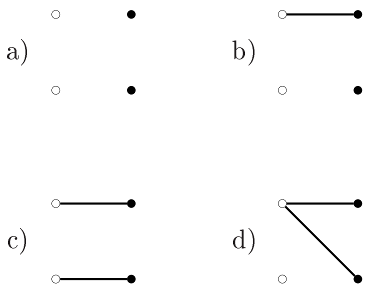

Figure 1

We reduce the question of whether G has a nontrivial automorphism to a reachability question in an undirected graph $f(G) = (V', E')$. V' contains one special node plus one node for each color,

$$V' = \{v_{id}\} \cup \{v^i \mid i \text{ is a color in } G\}.$$

The idea of the reduction is to place the edges in such a way that for every color i, node v_{id} is reachable from node v^i in $f(G)$ if and only if every automorphism in $\mathrm{Aut}(G)$ fixes the nodes of color i. For every pair (i, j) of colors, the edges between two color classes induce edges in E' in the following way:

Type of connection between i and j	Edges in E'
a)	None
b)	$(v^i, v_{id}), (v^j, v_{id})$
c)	(v^i, v^j)
d)	(v^i, v_{id}).

Also, if there is only one node of color i, the edge (v^i, v_{id}) is included in E'.

Lemma 2. *For each color i, one can reach node v_{id} from v^i in $f(G)$ if and only if every automorphism in* $\mathrm{Aut}(G)$ *fixes the nodes of color i.*

Proof. The proof from left to right is by induction on the number of edges in a path between v^i and v_{id}. If the path has length one, then the connections between the nodes of color i and some other color class in G must be like in cases b) or d). In any of these cases the nodes of color i must be fixed under any automorphism in G. If the path of minimal length has $k+1$ edges then there must be a color j and a path with k edges from v^j to v_{id} in $f(G)$. Also there must be an edge (v^i, v^j) in $f(G)$ which means that the connections between color classes i and j in G must be like in case c). By induction hypothesis every automorphism in $\mathrm{Aut}(G)$ fixes the nodes of color j, and the connection c) forces the nodes of color i to be fixed also.

For the other direction, suppose that for some color i node v_{id} is not reachable from v^i in $f(G)$. We will show that in this case there is some non-trivial automorphism in G. W.l.o.g. let $\{1, \ldots, k\}$ be the set of colors j such that v^j is connected to v^i in $f(G)$. Since v_{id} is not reachable from v^i, in case there are edges between two classes with colors in $\{1, \ldots, k\}$ in G, these edges form a connection of type c), and the only possible edges between classes of colors l and m, with $l \notin \{1, \ldots, k\}$ and $m \in \{1, \ldots, k\}$ are connections of type d). We claim that the color fixing permutation φ interchanging the two nodes of the classes of colors $j \in \{1, \ldots, k\}$ and fixing the rest of the nodes is a non-trivial automorphism in G. Let u, v be two nodes in V. If neither u nor v have colors in the set $\{1, \ldots, k\}$ then φ acts like the identity on (u, v). If both u and v have colors l, m in the set $\{1, \ldots, k\}$ then either there is no connection or the connections between the color classes l and m must be of type c). It follows that $(u, v) \in E$ if and only if $(u', v') \in E$, where u' and v' are the other nodes with colors l and m respectively. By the definition of φ, $(u', v') = (\varphi(u), \varphi(v))$. Finally, if u has color $l \notin \{1, \ldots, k\}$ and v has color $m \in \{1, \ldots, k\}$ and there is a connection between nodes of color l and m then it must be of type d) (a connection of type c) would force color l to be in $\{1, \ldots, k\}$). $(\varphi(u), \varphi(v)) = (u, v')$. As in the previous two cases we have $(u, v) \in E$ if and only if $(\varphi(u), \varphi(v)) \in E$. □ (Lemma 2)

Observe that Lemma 2 implies that the graph G has a nontrivial automorphism if and only if $f(G)$ has more than one connected component. □

3 Upper bounds for 3-GI and 3-GA

We deal in this section with graphs having up to 3 different nodes of each color. For a pair (G, H) of graphs, we will denote by $\mathrm{Iso}(G, H)$ the set of isomorphisms between both graphs. We denote by B_3 the set of 6 bijections defined over a domain set of 3 elements. An isomorphism between graphs with color classes of size 3 can be decomposed in a product of bijections from B_3.

Theorem 2. *The Graph Isomorphism problem restricted to graphs with color classes of size at most 3 is in the class* SL.

Proof. Let G and H be the input graphs. By Lemma 1 we can assume that for any pair (i, j) of colors there are at most 4 edges in each one of the input graphs having as endpoints nodes of colors i and j. If there are more than 4 edges we consider the dual connections. The three nodes of a color i in G are labeled i_1, i_2 and i_3, and in H by i'_1, i'_2 and i'_3.

We will reduce graphs G and H to a single undirected graph $f(G, H)$ and translate the isomorphism question to a reachability question in $f(G, H)$. For each color i we will consider 6 nodes in $f(G)$ corresponding to the possible bijections in B_3. Additionally there is one extra node w that will be used to indicate that some isomorphisms between subgraphs of G and H are not possible. The set of nodes in $f(G, H)$ is

$$\{v^i_\varphi \mid \varphi \in B_3 \text{ and } i \text{ is a color in } G\} \cup \{w\}.$$

The set of edges in $f(G, H)$ is constructed according to the following 2 rules. Observe that both rules can be applied in logarithmic space since the color classes have constant size 3 and there can only be at most 6 potential isomorphisms between the nodes of a given color.

Let C_i (resp. C'_i) denote the set of nodes of color i in G (resp. in H). For each color i the edges with both endpoints in C_i or C'_i might imply that some of the bijections from C_i to C'_i cannot be extended to an isomorphism between G and H. The restrictions in the set of possible bijections can also be induced by the connections with a different color class. In these cases we include some edges in the graph $f(G, H)$ as in Rule 1, indicating that a bijection between the nodes of a color cannot be extended to an isomorphism:

Rule 1: For every pair of colors i, j and every bijection $\varphi \in B_3$, we include in $f(G, H)$ the edge (v^i_φ, w) if the edges between C_i and C_j in G and the edges between C'_i and C'_j in H imply that no isomorphism in $\mathrm{Iso}(G, H)$ can map the nodes of C_i into C'_i like φ, that is, if for every $\psi \in B_3$, $\varphi \times \psi \notin \mathrm{Iso}(C_i \cup C_j, C'_i \cup C'_j)$, where $(C_i \cup C_j)$ is the graph made by the set of nodes C_i and C_j and the edges between them. For the trivial case in which there is only one color i in G, the edge (v^i_φ, w) is included in $f(G, H)$ when $\varphi \notin \mathrm{Iso}(C_i, C'_i)$.

We also include in $f(G, H)$ some edges between nodes corresponding to different color classes indicating that a partial isomorphism between the nodes of some color forces a partial isomorphism between the nodes of another color.

Rule 2: For every pair of colors i, j, and $\varphi \in B_3$. If for a pair of nodes $a, b \in \{i_1, i_2, i_3\}$ and a pair of nodes $a', b' \in \{i'_1, i'_2, i'_3\}$ and two bijections $\eta, \pi \in B_3$ the edges between the sets of nodes $\{i_a, i_b\}$ and $\{j_{\eta(a)}, j_{\eta(b)}\}$ are exactly the two edges $\{(i_a, j_{\eta(a)}), (i_b, j_{\eta(b)})\}$ and the edges between the sets of nodes $\{i'_a, i'_b\}$ and $\{j'_{\pi(a)}, j'_{\pi(b)}\}$ are exactly $\{(i'_a, j'_{\pi(a)}), (i'_b, j'_{\pi(b)})\}$ and $\varphi \times \psi \in \mathrm{Iso}(C_i \cup C_j, C'_i \cup C'_j)$ (for $\psi = \pi\varphi\eta^{-1}$) then we include in E' the edge (v^i_φ, v^j_ψ) (see Figure 2).

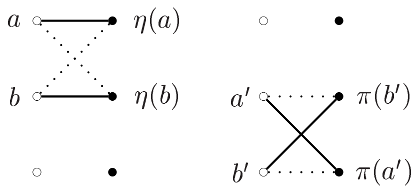

Figure 2: A situation in which Rule 2 is applied. The dotted lines indicate that these edges do not exist.

We show with the following lemmas that a reachability question in $f(G, H)$ can be used to decide whether G and H are isomorphic.

Lemma 3. *For a pair of colors i, j and $\varphi, \psi \in B_3$, if $(v^i_\varphi, w) \notin f(G, H)$ and $(v^j_\psi, w) \notin f(G, H)$ and $(v^i_\varphi, v^j_\pi) \notin f(G, H)$ for any $\pi \in B_3$, then $\varphi \times \psi$ is an isomorphism between the subgraphs $C_i \cup C_j$ and $C'_i \cup C'_j$.*

Proof. Let $G = (V, E)$ and $H = (W, E)$ be the input graphs and let us denote by a, b and c the nodes of color i in G, by d, e and f the color j nodes. We denote by the same symbols with ' the nodes of these colors in H. W.l.o.g we can suppose that for $l \in \{a, b, c\} \varphi(l) = l'$ and for $m \in \{d, e, f\}$ $\psi(m) = m'$. Let us suppose by contradiction that $\varphi \times \psi \notin \mathrm{Iso}(C_i \cup C_j, C'_i \cup C'_j)$. It follows that for a pair of nodes a, d,

$$(a, d) \in E \Leftrightarrow (\varphi(a), \psi(d)) \notin F.$$

We consider that $(a, d) \in E$, the other case is analogous. Since v^i_φ and v^j_ψ are not connected to w, by Rule 1 follows that there are two bijections η and $\pi \in B_3$ such that $\varphi \times \eta$ and $\pi \times \psi \in \mathrm{Iso}(C_i \cup C_j, C'_i \cup C'_j)$. Because of these facts we have that $\eta(d) \neq d'$ and $\pi(a) \neq (a')$. Again w.l.o.g. we can suppose $\eta(d) = e'$ and $\pi(a) = b'$. It follows $(\varphi(a), \eta(d)) = (a', e') \in F$ and $(\pi(a), \psi(d)) = (b', d') \in F$. The rest of the proof consists in considering the different possibilities for the bijections η and π showing that in each case we reach a contradiction.

Case 1: Suppose $\eta(d) = e', \eta(e) = d'$ and $\eta(f) = f'$. $(a', d') \notin F$ implies $(\varphi^{-1}(a'), \eta^{-1}(d')) = (a, e) \notin E$ and therefore $(\pi(a), \psi(e)) = (b', e') \notin F$ and also $(\varphi^{-1}(b'), \eta^{-1}(e') = (b, d) \notin E$. If $\pi(b) = a'$ then from $(a', e') \in F$ follows $(\pi^-1(a'), \psi^{-1}(e')) = (b, e) \in E$. But then we have that the edges between the nodes a, b, d, e in G and between their counterparts in H are exactly as is Rule 2, and there should be an edge in $f(G, H)$ from v^i_φ to some node of color j contradicting the hypothesis. The other possible situation in when $\pi(b) = c'$ and

$\pi(c) = a'$ but then from $(b,d) \notin F$ follows $(\varphi^{-1}\pi(b), \eta^{-1}\psi(d)) = (c,e) \notin E$, and from $(a',e') \in F$ follows $(\pi^{-1}(a'), \psi^{-1}(e')) = (c,e) \in E$ which is a contradiction.

Case 2: Suppose $\eta(d) = e', \eta(e) = f'$ and $\eta(f) = d'$. By the same arguments as in Case 1, we have $(b,f) \in E$, $(a,f) \notin E$ $(b',f') \notin F$ and $(b',e') \notin E$. If $\pi(b) = a'$ and $\pi(c) = c'$ then from $(b,e) \notin E$ follows $(\pi(b), \psi(e)) = (a',e') \notin F$ which is a contradiction. Finally, if $\pi(b) = c'$ and $\pi(c) = a'$ then it follows $(a,d), (b,f), (c,e) \in E$, $(a,f), (b,e), (c,d) \notin E$, $(a',e'), (b',d'), (c',f') \in F$ and $(a',d'), (b',f'), (c',e') \notin F$. Consider the pair of nodes a and e. If $(a,e) \in E$ then $(a',f'), (b',e') \in F$ contradicting the fact that there are at most 4 edges in H connecting the i and j nodes. On the other hand, if $(a,e) \notin E$ then the set of edges between the i and j color nodes are exactly $(a,d), (b,f), (c,e)$ in E and $(a',e'), (b',d'), (c',f')$ in F. By Rule 2 there should be some edge in $f(G,H)$ from v^i_φ to some node of color j, contradicting the hypothesis. □ (Lemma 3)

Lemma 4. *For each pair of colors i, j, and $\varphi, \psi \in B_3$, if there is a path from v^i_φ in $f(G,H)$ to v^j_ψ not having w as an intermediate node then every isomorphism in $\mathrm{Iso}(G,H)$ that maps the nodes of color i like φ, is forced to map the nodes of color j like ψ.*

Proof. We use induction on the length of a minimal path from v^i_φ to v^j_ψ in $f(G,H)$. If this path has length 1 then the colors i and j are different and the edge (v^i_φ, v^j_ψ) in $f(G,H)$ has been placed by Rule 2. This implies that for some $a, b \in \{1,2,3\}$ and some $c, d \in \{1,2,3\}$ the edges $(i_a, j_{\eta(a)})$, $(i_b, j_{\eta(b)})$ (for some $\eta \in B_3$) are the only edges between the sets of nodes $\{i_a,\, i_b\}$ and $\{j_{\eta(a)}, j_{\eta(b)}\}$ in G and the edges $(i_a, j_{\pi(a)})$, $(i_b, j_{\pi(b)})$ (for some $\pi \in B_3$) are the only edges between the sets of nodes $\{i'_c,\, i'_d\}$ and $\{j_{\pi(c)}, j_{\pi(d)}\}$ in H. Moreover, because of Rule 2, $\psi = \pi\varphi\eta^{-1}$. If an isomorphism in $\mathrm{Iso}(G,H)$ maps the i nodes to the i' nodes like φ, then for $l \in \{\eta(a), \eta(b)\}$ j_l must be mapped to a node j' in H connected to $i'_{\varphi\eta^{-1}(l)}$, and this node is $j_{\pi\varphi\eta^{-1}(l)} = j_{\psi(l)}$. It follows also for $c \notin \{a,b\}$, j_c is mapped to $j_{\pi\varphi\eta^{-1}(c)}$.

For the induction step, if the number of edges in the path from v^i_φ and v^j_ψ in $f(G,H)$ is $k+1$, let m be the first color after i in the path. There has to be some bijection $\phi \in B_3$ and an edge (v^i_φ, v^m_ϕ) between the i nodes and the j nodes in $f(G,H)$ and this edge must be introduced by Rule 2. By induction hypothesis, every isomorphism in $\mathrm{Iso}(G,H)$ mapping the i nodes like φ must map the m nodes like ϕ, and every isomorphism in $\mathrm{Iso}(G,H)$ mapping the m nodes like ϕ must map the j nodes like ψ. Both conditions together imply the result.

□ (Lemma 4)

Observe that Lemma 4 implies that for any color i, and $\varphi, \psi \in B_3$, if $\varphi \neq \psi$, and v^i_φ is reachable from v^i_ψ in $f(G,H)$ then there is no isomorphism in $\mathrm{Iso}(G,H)$ mapping the nodes of color i like φ. Another consequence of the lemma (together with the definition of Rule 1, is that for any color i and $\varphi \in B_3$, if node w can be reached from v^i_φ in $f(G,H)$ then there is no isomorphism in $\mathrm{Iso}(G,H)$ mapping the nodes of color i like φ.

Lemma 5. *Suppose that there are k different colors in G and H. If there is a set of nodes in $f(G,H)$, $v^1_{\varphi_1}, \ldots, v^k_{\varphi_k}$ one of each color, and such that no other node in $f(G,H)$ is reachable from this set then $\varphi_1 \times \ldots \times \varphi_k$ is an isomorphism between G and H.*

Proof. The proof is by induction on the number of color classes k in G and H. If there is only one color the result is trivial. For the case of two colors i and j, consider that from the set of nodes v^i_φ and v^j_ψ one cannot reach any other node in $f(G,H)$. This implies that the only possible edge in $f(G,H)$ with an endpoint in this set is the one connecting both nodes. If this edge does not exist the result follows by Lemma 3. On the other hand, if (v^i_φ, v^j_ψ) is an edge in $f(G,H)$ then the edge was placed by Rule 2 and $\varphi \times \psi \in \mathrm{Iso}(G,H)$.

For the induction step, consider that there are k colors in G and H and there is a set of nodes one of each color $v^1_{\varphi_1}, \ldots, v^k_{\varphi_k}$ in $f(G,H)$. Consider the graphs G' and H' obtained by deleting the nodes of color k in G and H and all the edges having an endpoint of this color. Since eliminating one color can only reduce the set of local restrictions for isomorphisms, there is no new edge in $f(G',H')$ that was not already present in $f(G,H)$ and therefore from $v^1_{\varphi_1}, \ldots, v^k_{\varphi_{k-1}}$ no other node is reachable in $f(G',H')$. By induction hypothesis $\varphi_1 \times \ldots \times \varphi_{k-1} \in \mathrm{Iso}(G',H')$. We claim that this isomorphism between G' and H' can be extended to an isomorphism in $\mathrm{Iso}(G,H)$ by mapping the nodes in C_k to C'_k as in φ_k.

To see that this is an isomorphism we will show that for every $j < k$ it holds $\varphi_k \times \varphi_j \in \mathrm{Iso}(C_k \cup C_j, C'_k \cup C'_j))$. If the edge $(v^k_{\varphi_k}, v^j_{\varphi_j})$ belongs to $f(G,H)$ then the edge was placed by Rule 2 and $\varphi_k \times \varphi_j \in \mathrm{Iso}(C_k \cup C_j, C'_k \cup C'_j))$. On the other hand if this edge does not exist, there is no other edge in $f(G,H)$ between k and j nodes having $v^k_{\varphi_k}$ or $v^j_{\varphi_j}$ as endpoint (from the set $v^1_{\varphi_1}, \ldots, v^k_{\varphi_k}$ no other node can be reached). The result follows then by Lemma 3.

□ (Lemma 5)

It follows by Lemmas 4 and 5 that there is an isomorphism from G to H if and only if there is a set of nodes in the graph $f(G,H)$, one of each color, such that from this set no other node in $f(G,H)$ can be reached. In order to transform this into a question in SL we need the following lemma:

Lemma 6. *Let A and B be two connected components in $f(G,H)$ satisfying the following conditions:*

i) The nodes in A have different colors and the nodes in B have different colors.
ii) Node w does not belong to $A \cup B$ and.
iii) The intersection of colors in A and B is not empty.

Then the set of colors present in A is the same as the set of colors present in B.

Proof. We show that for any pair of colors i and j, and for two bijections φ and $\xi \in B_3$, if the node w in $f(G,H)$ cannot be reached neither from v^i_φ nor from v^i_ξ and if v^i_φ has a neighbor of color j in $f(G,H)$, then so does v^i_ξ. The result follows from this fact.

Suppose that (v^i_φ, v^j_ψ) is an edge in $f(G,H)$. This edge was set by Rule 2 and therefore there is a pair of nodes $a, b \in \{i_1, i_2, i_3\}$ and a pair of nodes $a', b' \in \{i'_1, i'_2, i'_3\}$ and two bijections $\eta, \pi \in B_3$ satisfying the conditions of Rule 2 and such that $\psi = \pi\varphi\eta^{-1}$. Since v^i_ξ is not connected to w, there must be a bijection $\gamma \in B_3$ satisfying $\xi \times \gamma \in \mathrm{Iso}(C_i \cup C_j, C'_i \cup C'_j)$. η and π describe the connections between i and j nodes in G and H and therefore we have $\gamma = \pi\xi\eta^{-1}$. We are again in the conditions of Rule 2, and the edge (v^i_ξ, v^j_γ) belongs to $f(G,H)$. □ (Lemma 6)

We know that there is an isomorphism from G to H if and only if there is a set of nodes in $f(G,H)$, one of each color and such that from this set no other node in $f(G,H)$ can be reached. By Lemma 6 in order to test this it suffices to check for each color i that there is one node v^i_φ in $f(G,H)$ from which neither w nor two nodes of the same color can be reached (this question can be solved within the class SL). In order to see this observe that if G and H are isomorphic, such a set must exist. By Lemma 6 if such a set exists then there is a set of color disjoint connected components in $f(G,H)$ containing all colors. □

We show now that the graph automorphism problem for colored graphs with color classes of size 3 lies also in the class SL. Although a direct proof similar to the one in Theorem 2 is possible, it is easier to give a reduction from 3-GA to UGAP based on the fact that 3-GI $\in$ SL.

Theorem 3. *The Graph Automorphism problem restricted to graphs with color classes of size at most 3 is in the class* SL.

Proof. We will show that 3-GA is logarithmic space many-one reducible to UGAP. This implies that 3-GA lies in SL. Let $G = (V, E)$ be a graph with its nodes partitioned into color classes of size at most three. We denote by $G_{[i]}$ a copy of G but with node i marked with a new special color. There is a non-trivial automorphism in G if and only if for a pair of distinct nodes of the same color i and j in V, there is an automorphism mapping i to j, if and only if for such a pair of nodes $G_{[i]}$ is isomorphic to $G_{[j]}$. Since the color classes in $G_{[i]}$ have size 3 at most, this means that 3-GA is reducible to a set of disjunctive queries to 3-GI. By Theorem 2, 3-GI lies in SL and can be reduced to UGAP. The list of queries to 3-GI can then be reduced to a list of reachability queries in undirected graphs. The disjunctive list of queries to UGAP can be reduced to a single one by connecting the graphs in parallel. This provides a many-one reduction from 3-GA to UGAP. □

4 Lower bounds

We prove now the hardness results for 2-GI and 2-GA.

Theorem 4. 2-GI *is hard for* SL *under logarithmic space many-one reductions.*

Proof. We show that the graph accessibility problem for undirected graphs, UGAP, is reducible to the complement of 2-GI. The result follows since UGAP is logarithmic space complete for SL, and this class is closed under complementation [16].

Let $G = (V, E)$ be an undirected graph with two designated nodes $s, t \in V$. Consider the new graph $G' = G_1 \cup G_2$ where G_1 and G_2 are two copies of G, and for a node $v \in V$ let us call v_1 and v_2 the copies of v in G_1 and G_2 respectively. Furthermore, we color each pair of nodes v_1, v_2 with color i_v, and color t_1 with a special color 1 and t_2 with another color 2. We claim that there is no path from s to t in G if and only if there is automorphism φ in G' mapping s_1 to s_2. Clearly if there is no path between s and t in G, these two nodes belong to different connected components. The desired automorphism can be obtained by mapping the nodes of the connected component of s_1 in G_1 to the corresponding nodes in G_2 and mapping the rest of the nodes in G' (and in particular t_1) to themselves.

Conversely, if there is a path between s and t in G, the mentioned automorphism φ does not exist since the nodes s_2 ($\varphi(s_1)$) and t_1 ($\varphi(t_1)$) should be in the same connected component, but there are no edges between G_1 and G_2 in G'.

The question of whether there is an automorphism in G' with the mentioned properties, can in turn be reduced to 2-GI. Let H_1 be a copy of G' with node s_1 having a special color 3, and H_2 be another copy with s_2 having color 3. There is an automorphism mapping s_1 to s_2 in G' if and only if H_1 and H_2 are isomorphic. The size of the color classes in each of the graphs H_1 and H_2 is at most 2. □

Corollary 1 (of the proof). 2-GA *is hard for* $\overline{\text{UCC}}$ *under logarithmic space many-one reductions.*

Proof. The reduction for this result is the same as in the previous theorem. Observe that in G' the color classes are of size at most 2 and there is a nontrivial automorphism in G' if and only if G has more than a single connected component. □

5 Conclusions

The following results follow directly from the upper and lower bounds given in Sections 2, 3 and 4.

- 2-GA is equivalent to $\overline{\text{UCC}}$ under logarithmic space many-one reductions.
- 2-GI and 3-GI are complete for SL under logarithmic space many-one reductions.
- 3-GA belongs to SL.

A natural way to continue this research is to study the situation for other bounds $b \geq 4$ on the size of the color classes. From the results in [17] it can easily be derived that for $b \geq 2$, b^2-GI is hard for the modular class $\mathrm{Mod}_b\mathrm{L}$, and $2b^2$-GA is also hard for $\mathrm{Mod}_b\mathrm{L}$. From this follows that 4-GI is hard for $\oplus\mathrm{L}$ and therefore a proof of the fact that 4-GI belongs to SL would imply that $\oplus\mathrm{L}$ is included in SL which is something we do not expect. It remains an interesting open problem whether better upper bounds than the ones given in [15] can be obtained for b-GI and b-GA for other special values of b ($b \geq 4$).

We observe that the blow-up in the complexity of the problem when going from color classes of size 3 to size 4 also happens in the related area of graph identification using first-order formulas with counting. Immerman and Lander show in [8] that 3 variables suffice to identify all colored graphs of color size 3, while $\Omega(n)$ variables are needed to identify all graphs of color size 4 using first order formulas with counting, as proved in [5].

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