Algorithms and Data Structures

Optimal Search Trees; Tries

Ulf Leser

Content of this Lecture

- Optimal Search Trees
  - Definition
  - Construction
  - Analysis
- Searching Strings: Tries

Static Key Sets, Varying Access Frequencies

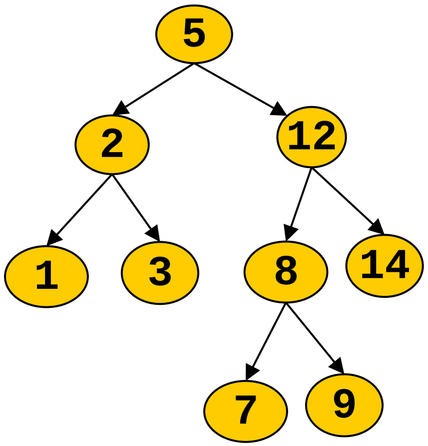
- Sometimes, the set of keys is “fixed”
  - Streets of a city, cities in a country, keywords of a prog. lang., ...
- Often, searches are much more frequent than updates
  - We may spent more effort for reorganizing the tree after updates
- Example: Large-scale web search engines
  - Recall: A search engine creates a dictionary; every word has a link to the set of documents containing it
  - The dictionary must be accessed very fast, changes are rare
  - Often, engines build complex structures to optimally support searching over the current set of documents considered as static
    - Defer updates: Changes are buffered and bulk-inserted periodically
    - Search either searches two data structures, or misses are accepted

Scenario

- Assume a set K of keys and a bag R of requests (workload)
  - Every request searches a k∈K; k's may appear multiple times in R
  - In contrast to SOL, we now don't care about the order of requests
  - Like SOL with fixed access frequencies – but now we consider trees
- Naïve approach
  - Build an AVL tree over K
  - Every r∈R costs O(log(|K|)), i.e., we need O(|R|*log(|K|))
  - This is optimal, if every k∈K appears with the same frequency in R
- What if R is highly skewed?
  - Skewed: k's are not equally distributed in R
  - Rather the norm than the exception in real life (Zipf, ...)
  - In contrast to SOL, finding an optimal search tree for R is not trivial

Example

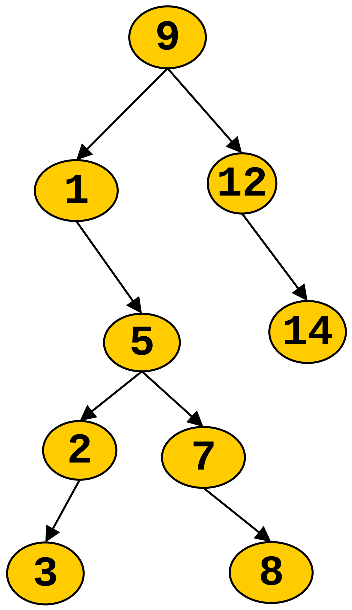
- K={1,2,3,5,7,8,9,12,14}
- We build an AVL tree



- R_1={2,5,8,7,3,12,1,8,8}
  – 2+1+3+4+3+2+3+3+3=31 comparisons
- R_2={9,9,1,9,2,9,5,3,9,1}
  – 4+4+3+4+2+4+1+3+4+3=32 comparisons

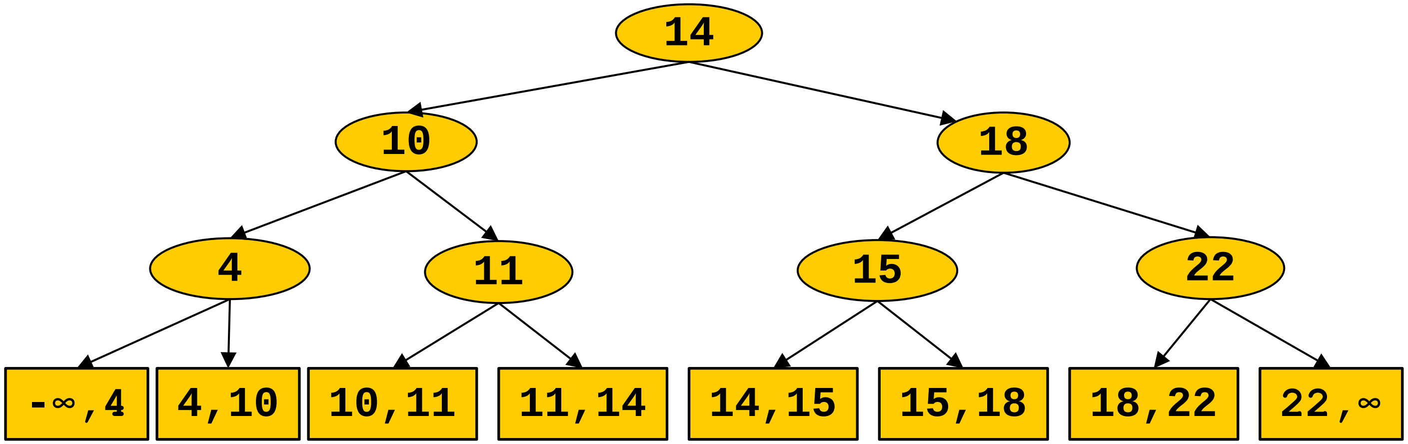
Example

- Let's optimize the tree for R_2
  - Not a AVL tree any more
- R_2={9,9,1,9,2,9,5,3,9,1}
  ={9,9,9,9,9,1,1,2,5,3}
  - 9 and 1 should be high in the tree
  - 1+1+1+1+1+2+2+4+3+5=21
    - Versus 32
- Not good for R_1
  - R_1={2,5,8,7,3,12,1,8,8}
  - 4+3+5+4+5+2+2+5+5=35
    - Versus 31
- Is this truly the optimal search tree for R_2?



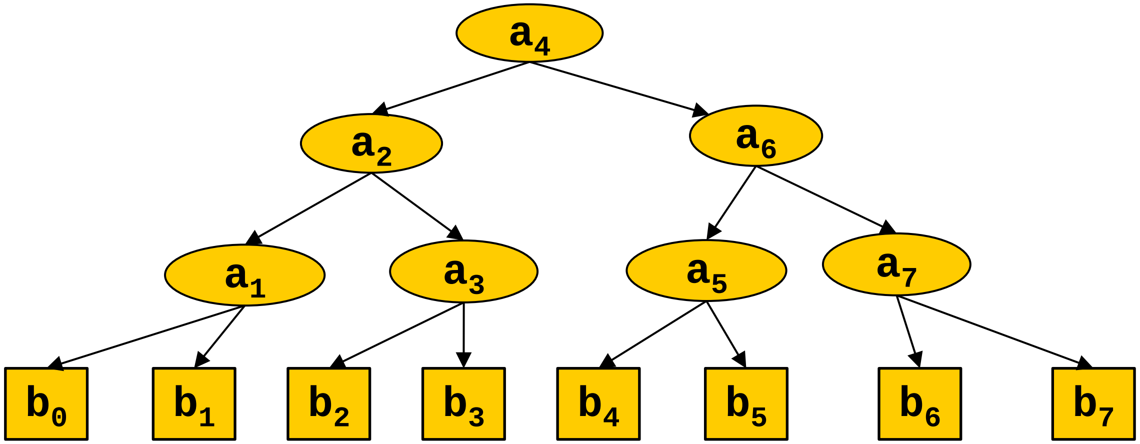
Request Model

- Assume an (ordered) set K of keys, $K=\{k_1, k_2, ..., k_n\}$
- Every k is searched with frequency $a_1, a_2, ..., a_n$
- No-key intervals $]-\infty,k_1[$, $]k_1,k_2[$, ..., $]k_{n-1},k_n[$, $]k_n,+\infty[$ are searched with frequencies $b_0, b_1, ..., b_n$
  - We need to consider costs of searches that fail
- Together: $R=\{a_1, a_2, ..., a_n, b_0, b_1, ..., b_n\}$



Request Model

- Assume an (ordered) set K of keys, $K=\{k_1, k_2, \ldots, k_n\}$
- Every k is searched with frequency $a_1, a_2, \ldots, a_n$
- No-key intervals $]-\infty,k_1[$, $]k_1,k_2[$, ..., $]k_{n-1},k_n[$, $]k_n,+\infty[$ are searched with frequencies $b_0, b_1, \ldots, b_n$
  - We need to consider costs of searches that fail
- Together: $R=\{a_1, a_2, \ldots, a_n, b_0, b_1, \ldots, b_n\}$



Optimal Search Trees

- Definition
  Let T be a search tree for K and R a workload. The cost P(T) of T for R is defined as

$$P(T)=\sum_{i=1}^{n}\left(depth(k_i)+1\right)*a_i+\sum_{j=0}^{n}\left(depth(\,]k_j,k_{j+1}[\,)+1\right)*b_j$$

- Definition
  Let K be a set of keys and R a workload. A search tree T over K is optimal for R iff

$$P(T)=\min\left\{P(T')\mid T'\ is\ search\ tree\ \ for\ \ K\right\}$$

One More Definition

- Definition
  Let T be a search tree over K and R a workload. The *weight W(T) of T for R* *is:*

$$W(T)=\sum_{i=1}^{n} a_i + \sum_{j=0}^{n} b_j$$

- Thus, the weight of T is simply |R|
- We will need this definition for subtrees

Content of this Lecture

- Optimal Search Trees
  - Definition
  - Construction
  - Analysis
- Searching Strings: Tries

Finding the Optimal Search Tree

- Bad news: There are exponentially many search trees
  - We cannot enumerate all search trees, compute their cost, and then choose the cheapest
  - Proof omitted
- Good news: We don't need to look at all possible search trees
  - We can use a divide & conquer approach
  - Dynamic programming: Build large solutions from smaller ones
    - Recall max_subarray etc.
    - Here: Build larger optimal search trees from smaller optimal STs

General Idea

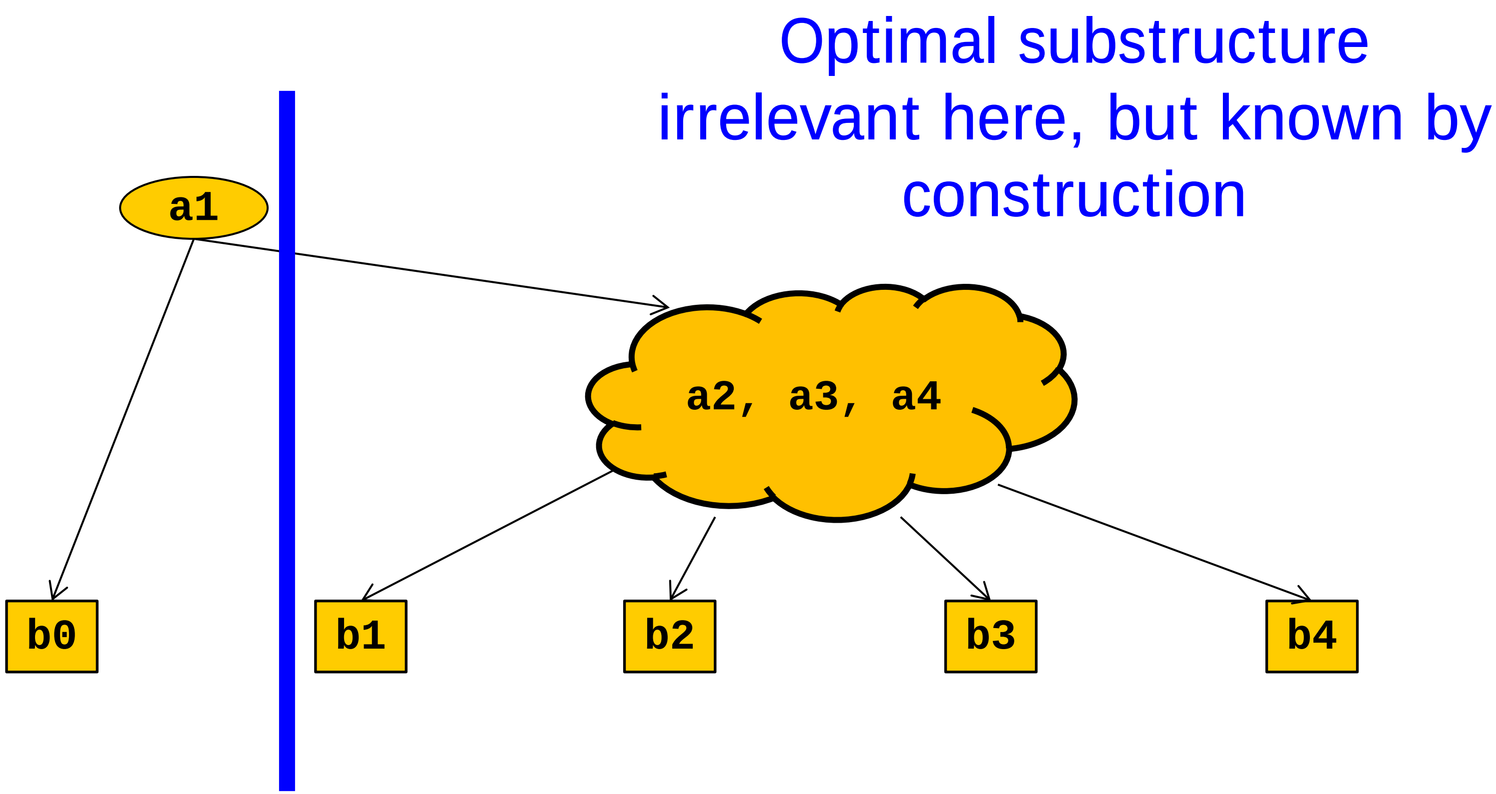
- Observation: We can define P(T) recursively
  - Let k_r be root of T and T_{lr}=leftChild(k_r), T_{rr}=rightChild(k_r)
    - "lr: Left-of-r"; "rr: Right-of-r"
  - Clearly: $P(T) = P(T_{lr}) + P(T_{rr}) + a_r + W(T_l) + W(T_{rr})$
    $= P(T_{lr}) + P(T_{rr}) + W(T)$
  - Since W(T) is the same for every possible search tree, the cost of a tree only depends on the cost of its subtrees
- Problem: We do not know k_r, but we need to find it
  - k_r divides T into a left part (T_{lr}) and a right part (T_{rr})
  - Both T_{lr} and T_{rr} are smaller than T
  - Assume we knew $P(T_{lr})$ and $P(T_{rr})$ for every possible k_r
    - Both are smaller, so we can compute T_l/T_r values bottom-up
  - We can test all n different k_r's and find the one maximizing the term $P(T_{lr}) + P(T_{rr}) + W(T)$

Example

- We want to compute the optimal search tree T for the keys a1-a4 and no-key ranges b0-b5
- One of the keys a1, a2, a3, a4, must be the root

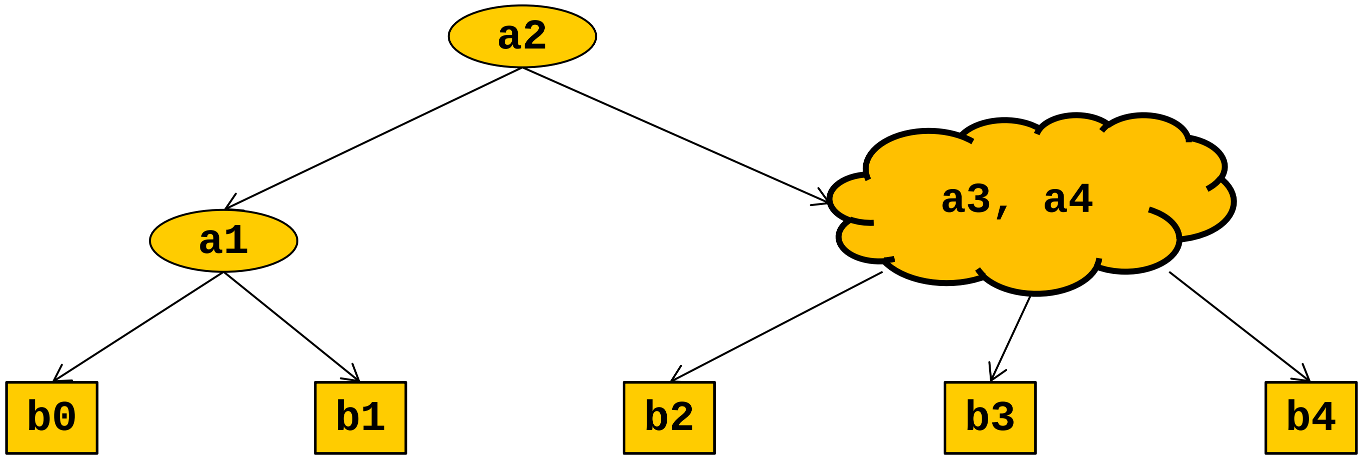
Example Continued

- If a1 would be the "optimal root", the cost of P(T) would be P(b2)+P(b1...b4)+W(T)



Example Continued

- If a2 would be the "optimal root", the cost of P(T) would be P(b0..b1)+P(b2..b4)+W(T)



Formal: A Divide & Conquer Approach

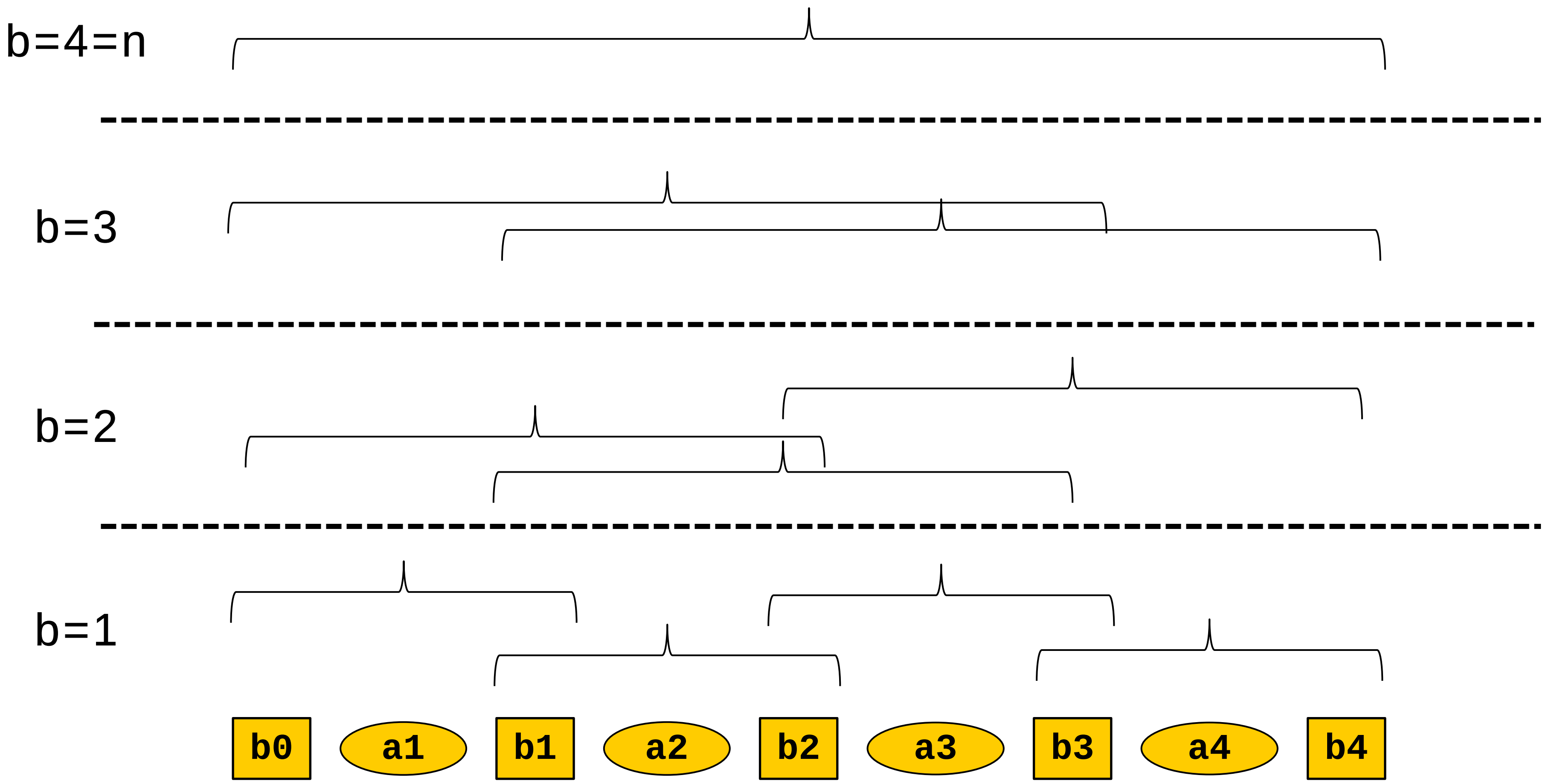
- Consider a range R(i,j) of keys and intervals
  - $R(i,j) = \{]k_i,k_{i+1}[, k_{i+1},]k_{i+1},k_{i+2}[, k_{i+2}, \ldots k_j,]k_j,k_{j+1}[\}$
- Assume that R(i,j) is represented as subtree T(i,j) of T(1,n)
  - That's not the case in all topologies for T; the "left" part of R could lie in a different subtree than the "right" part
- One of the $k_r \in R(i,j)$ must be the root of this subtree
- Thus, k_r divides R(i,j) in two halves R(i,r-1), R(r,j)
- Assume we know the optimal trees for all sub-ranges R(i,i+1), R(i,i+2), ..., R(i,j-1), R(i+1,j), ..., R(j-1, j)
- Then, we find the r creating the optimal tree T(i,j) using

$$P(T(i,j)) = W(T(i,j)) + \min_{r=i+1..j} \Big(P\big(T(i,r-1)\big) + P\big(T(r,j)\big) \Big)$$

Bottom-Up Computation

- We systematically enumerate smaller R(i,j) and puzzle them together to larger ones
- Let P(i,j) be the cost of the optimal search tree for R(i,j)
- To compute P(i,j), we (1) need the P and W-values of all possible enclosed subtrees and we (2) need to find the optimal value of r
- We perform induction over the breadth b of intervals: All intervals of breadth 0, 2 ... n (and we are done)
  – Breadth of an interval: Number of keys contained

Illustration



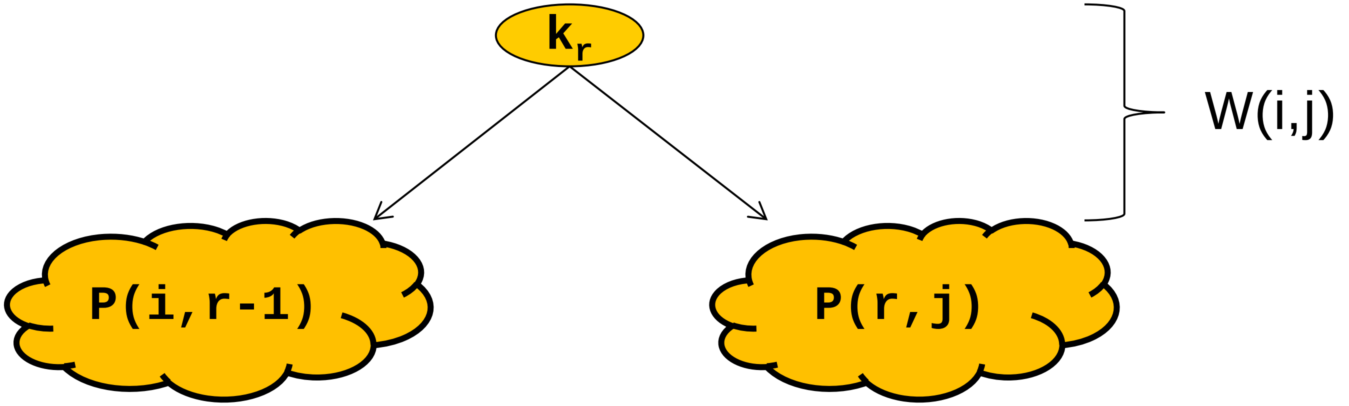
Induction Start

- $b=0$; all subintervals (i,i)
  - This is a leaf (an interval without keys), no root selection required
  - $\forall 0 \le i < n+1$: $W(i,i) = b_i$
    $P(i,i) = W(i,i)$
- $b=1$; all subintervals $(i,i+1)$
  - The root is always k_{i+1}
    - The only key in this interval; $l=i+1$
  - $\forall 0 \le i < n$: $W(i,i+1) = b_i + a_{i+1} + b_{i+1}$
    $P(i,i+1) = P(i,i) + W(i,i+1) + P(i+1,i+1)$

Induction

- General case: $b>1$, subintervals (i,j) with $j-i=b>1$
  - Induction hypothesis: We know W, P for all intervals of breadth<b
  - Find the index r for the optimal root of the subtrees
  - Then compute: $W(i,j) = W(i,r-1) + a_l + W(r,j)$

    $P(i,j) = P(i,r-1) + W(i,j) + P(r,j)$

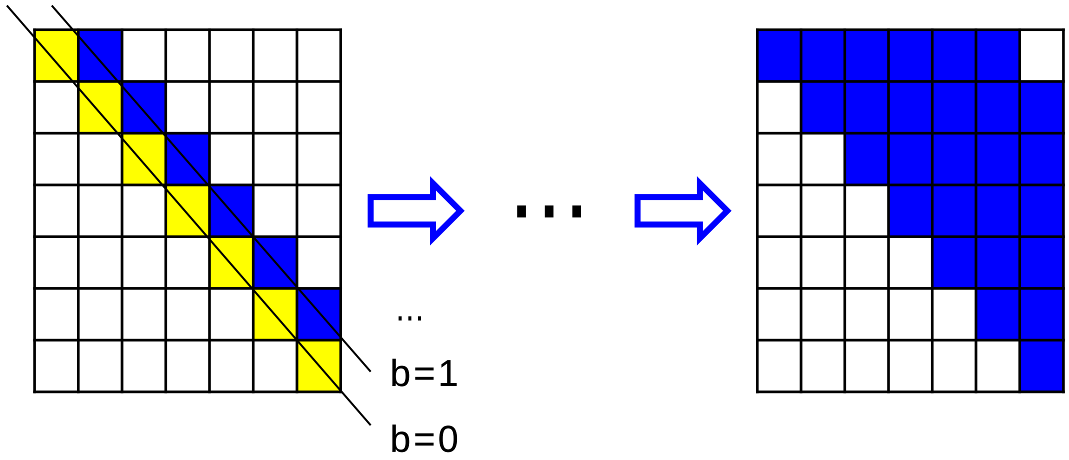


Content of this Lecture

- Optimal Search Trees
  - Definition
  - Construction
  - Analysis
- Searching Strings: Tries

Implementation

- There are only (n+1)*(n+1) different pairs i,j
- We essentially fill a quadratic matrix of size (n+1)*(n+1) for W and one for P
  - Since j≥i, we actually only need half of each matrix
- Both matrixes are iteratively filled from the main diagonal to the upper-right corner



Analysis

- Space
  - We need 2 arrays of size O(n*n)
  - Space complexity: $O(n^2)$
- Time
  - Cases b=0 and b=1 are O(n)
  - We enumerate breadths from 2 to n
  - For each b, we consider all possible start positions: O(n-b) many
  - In each range, we need to find the optimal l – this is O(b)
  - A range has max size n-1
  - Together: $O(n^3)$

```
initialize W(i,i);
initialize P(i,i);
initialize W(i,i+1);
initialize P(i,i+1);
for b = 2 to n do
  for i = 0 to (n-b) do
    j := i+b;
    find optimal l in [i,j];
    W(i,j) := …
    P(i,j) := …
  end for;
end for;
```

Constructing the tree

- We only showed how to compute the cost of the optimal tree, but not how to build the tree itself
- But this is simple since we never revise decisions
- We can “grow” the tree whenever we have computed a new optimal root l
- For instance, we can define a r(i,j):=l in every step; the sequence of computed l-values fully determine the tree

Relevance

- Nice and instructive
- Runtime can actually be reduced to $O(n^2)$
- But: $O(n^2)$ is still quite expensive for large n
- Fortunately, one can compute „almost“ optimal search trees in linear time
  - Not this lecture

Content of this Lecture

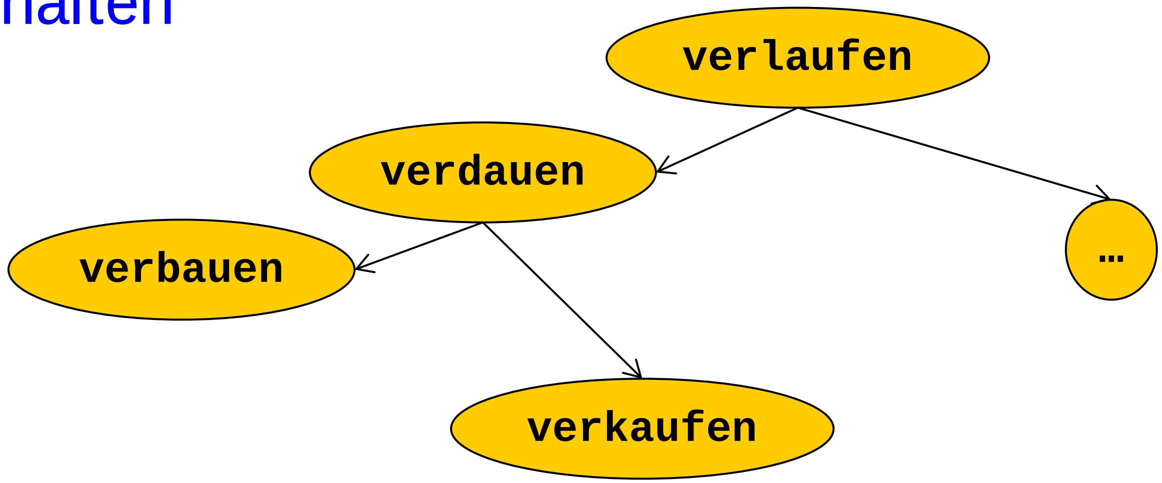
- Optimal Search Trees
- Searching Strings: Tries

Keys that are Strings

- Assume K is a set of strings of maximal length m
- We can build an AVL tree over K
- Searching requires O(log(n)) key comparisons
- But: Each string-comp requires m char-comps in WC
  - Very pessimistic, but we do WC analysis
- Together: We need O(|k|*log(n)) character comparisons for searching a key k
- Observation
  - "Similar" strings will be close neighbors in the tree
  - These will share prefixes (the longer, the more similar)
  - These prefixes are compared again and again

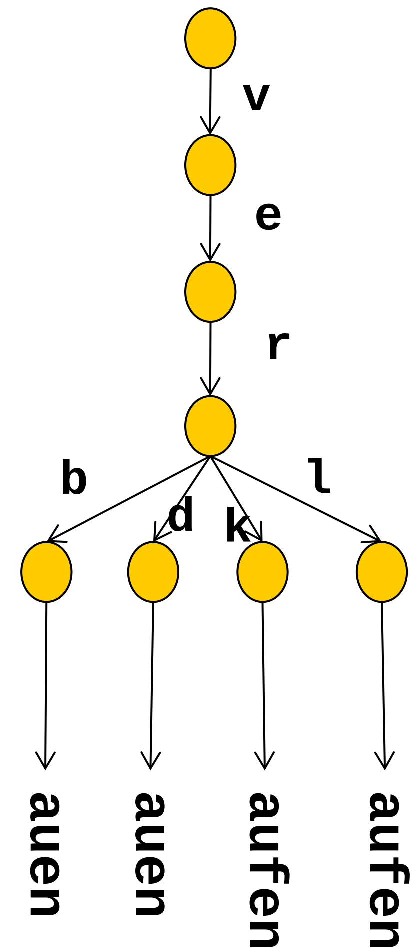
Example

k=„verhalten“



Tries

- Tries are edge-labeled trees of order |Σ|
  - Developed for Information Retrieval
- Edges are labeled with chars from Σ
- Idea: Common prefixes of keys are represented only once
- Problem: If "verl" is a key?
  - Trick: Add a "$" (not in Σ) to every string
  - Then every and only leaves represent keys



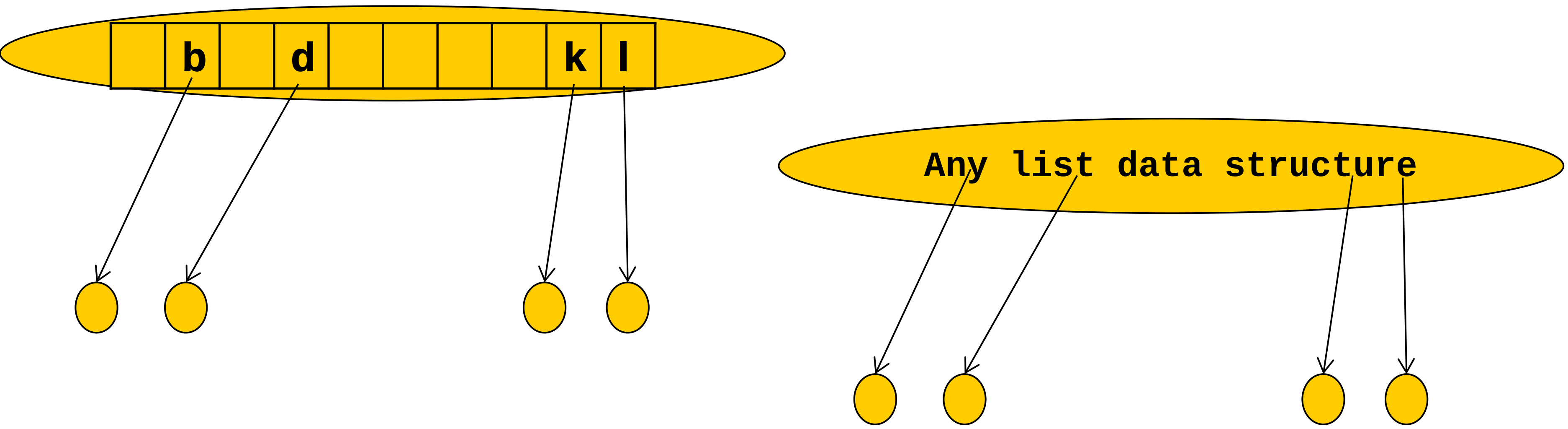
Analysis

- Construction of a trie over K?
  - Let len(K) be the sum of all key lengths in K
  - We start with an empty tree and iteratively add all $k \in K$
  - To add a key k, we char-match k in the tree as long as possible
  - As soon as no continuation is found, we build a new branch
  - This requires O(|k|) operations (char-comps or node creations)
  - It follows: Construction is in O(len(K))
- Searching a key k (which maybe in K or not in K)
  - We match k from root down the tree
  - When k is exhausted and we are in a leaf: $k \in K$
  - If no continuation is found or we end in an inner node: $k \notin K$
  - It follows: Searching is in O(|k|)
  - But ...

Space Complexity

- We have at most len(K) edges and len(K)+1 nodes
  - Shared prefixes make the actual number smaller
- But we also need pointer to children
- To achieve our search complexity, choosing the right pointer must be in O(1)
- This adds O(len(K)*|Σ|) pointers
- Too much for any non-trivial alphabet
  - Digital tries are a popular data structure in coding theory
  - There, |Σ|=2, so the pointers don't matter much
  - But beware – the trees get very deep
- Furthermore, most of the pointers will be null
  - Depending on |Σ|, |K|, and lengths of shared prefixes

Alternatives

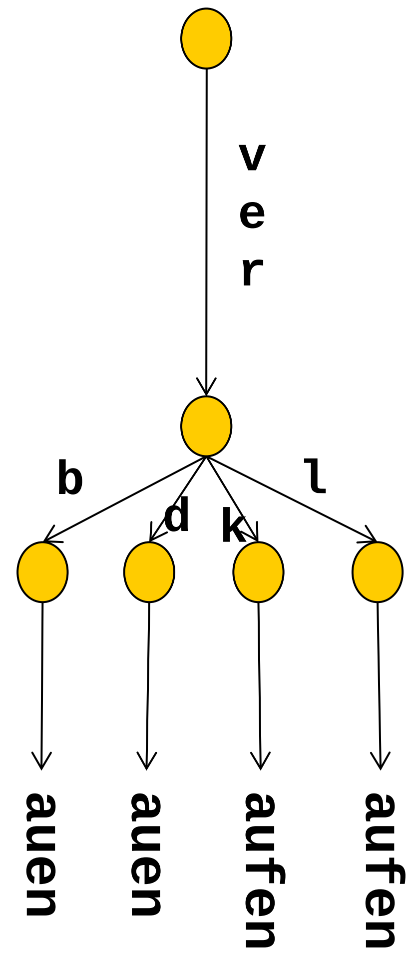


- Full array for children ptr
- Advantage: O(|k|) search
- Disadvantage: Excessive space consumption

- Dense array for children ptr
- Advantage: O(len(K)) space
- Disadvantage: Search is O(|k|*log(|Σ|))

Compressed Tries = Patricia Trees

- We can save further space
- A patricia tree (or radix tree) is a trie where edges are labeled with (sub-)strings, not with characters
- All sequences S=<node, edge> which do not branch are compressed into a single edge labeled with the concatenation of the labels in S
- More compact, less pointer
- Slightly more complicated implementation
  - E.g. insert requires splitting of labels



Exemplary Questions

- Recall the definition of a trie. Give in implementation (in pseudo code) for (a) searching a key k and (b) building a trie for a string set K. You may presuppose a data structure „list“ with operations add(c, p) for adding a pair of character and pointer and retrieve(c), which returns the pointer associated to c or nil.
- Build an optimal search tree for K={5,12,15,20} and R={6,2,3,8,11,5,2,1,4}. Show the complete tables for W and P
- Prove that all tries for any permutation of a set of strings are identical