Algorithms and Data Structures

(Search) Trees

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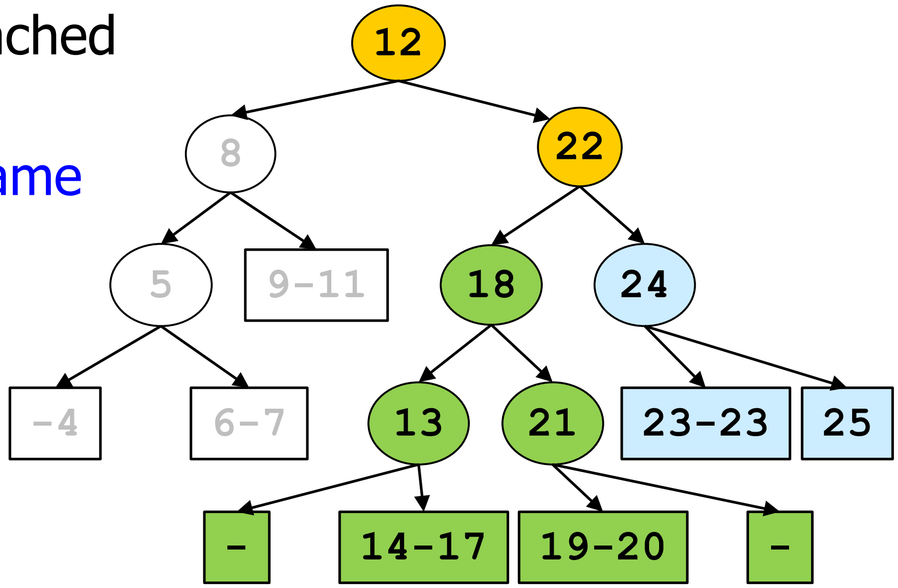
Source: whmsoft.net/

Content of this Lecture

- Trees
- Search Trees
- Natural Trees

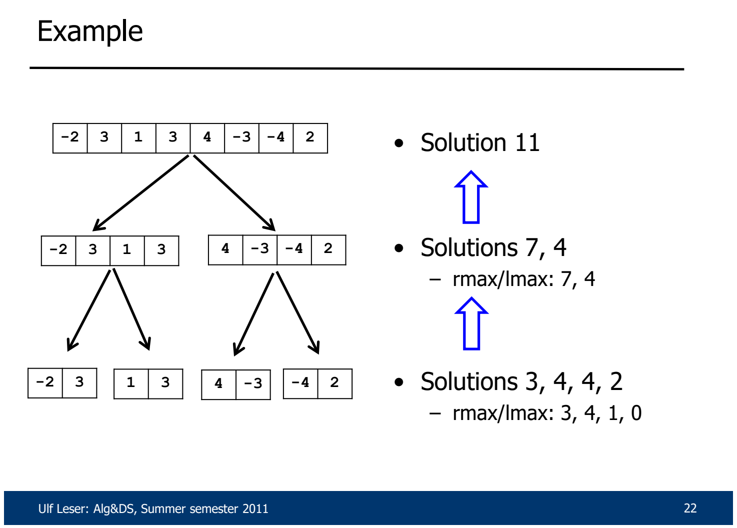
Motivation

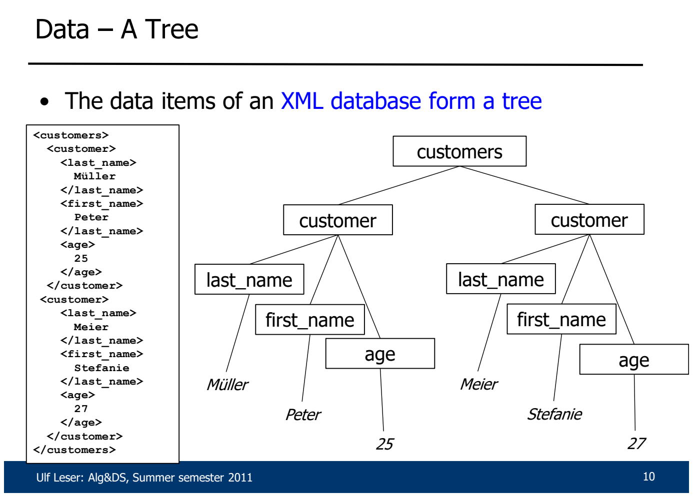
- In a list, (almost) every element has one predecessor / successor
- In a tree, (almost) every element has one predecessor but many successors
- These splits partition the set of all elements of the list
  - Every node in a tree can be reached by only one path from root
  - Partitions: All nodes with the same prefix in their access paths
  - Prominent split criterion: Order
    - Elements with lower rank to left subtree, with higher rank to the right subtree



Trees are everywhere in computer science

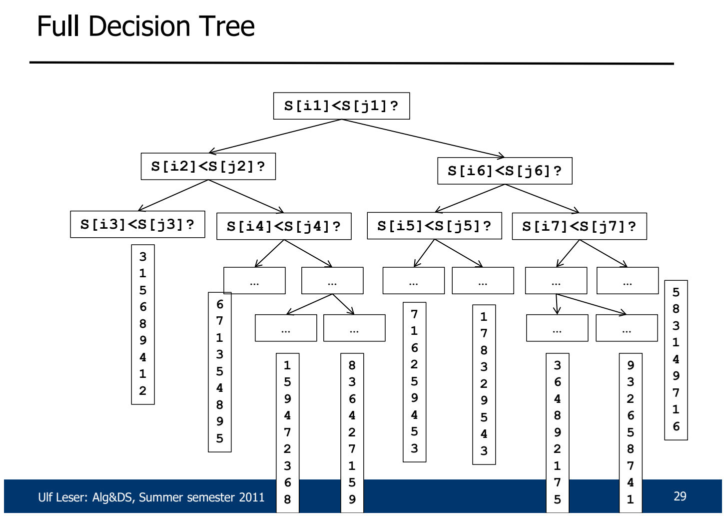
- Divide-and-conquer call stacks
  - Max-subarray
  - Merge-Sort
  - QuickSort
  - …
- XML
  - depth-first vs breadth-first traversal



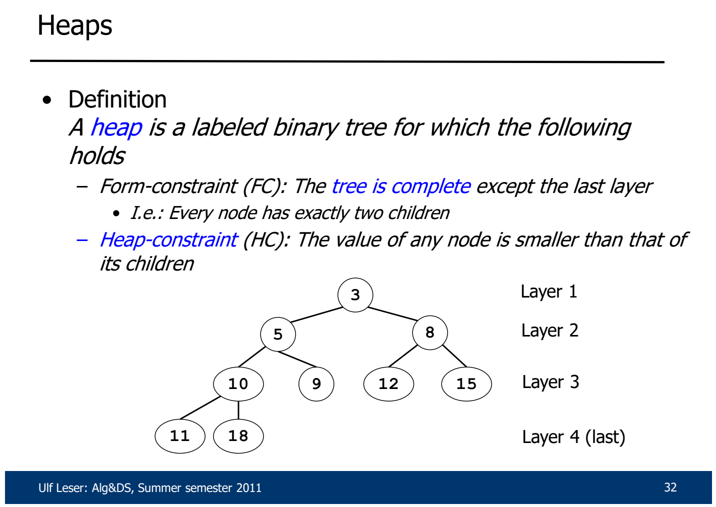


Already Seen

- Decision trees for proving the lower bound for sorting



- Heaps for priority queues



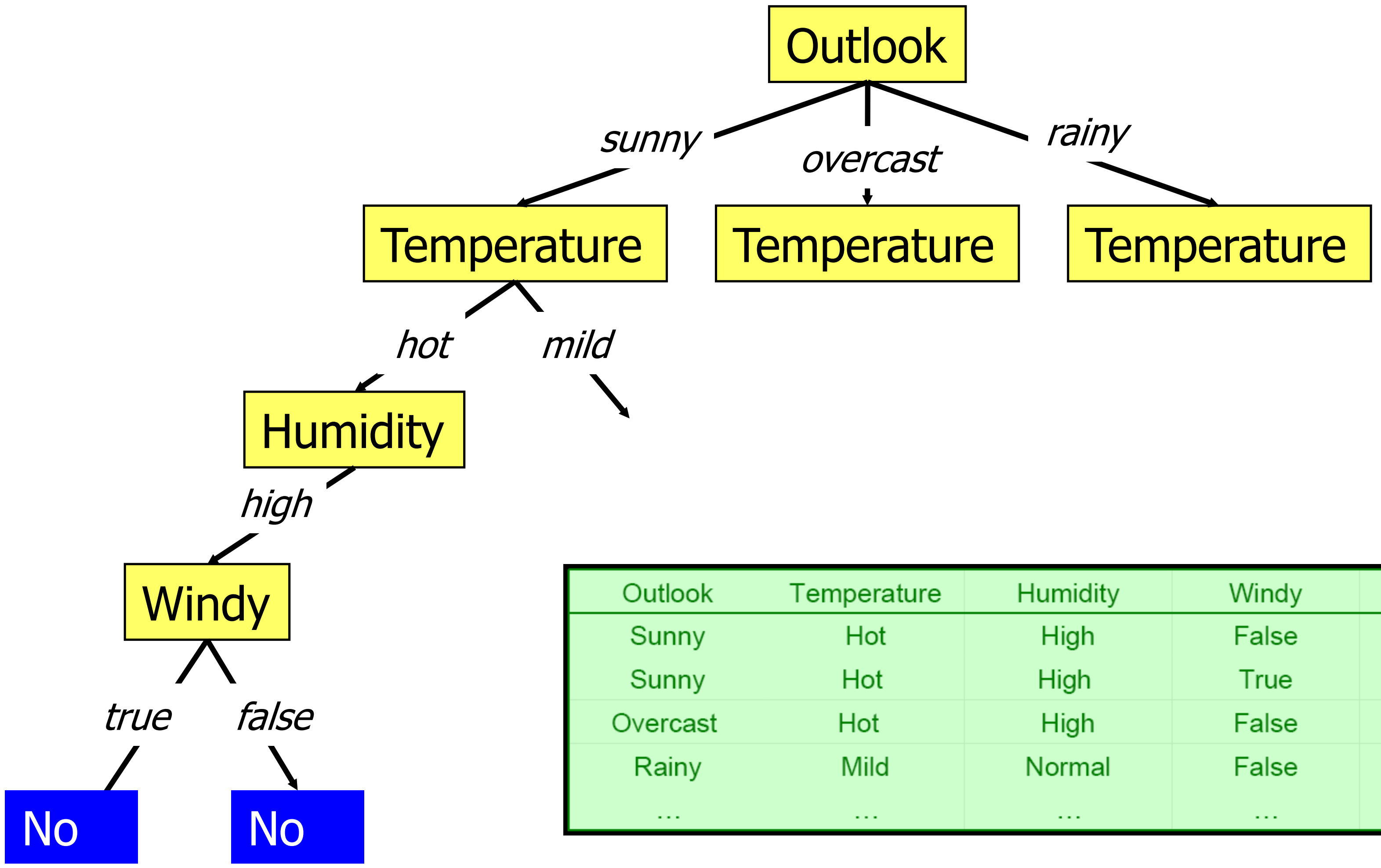
- …

Machine Learning

- Want to go to a football game?
- Might be canceled – depends on the whether
- Let‵s learn from examples

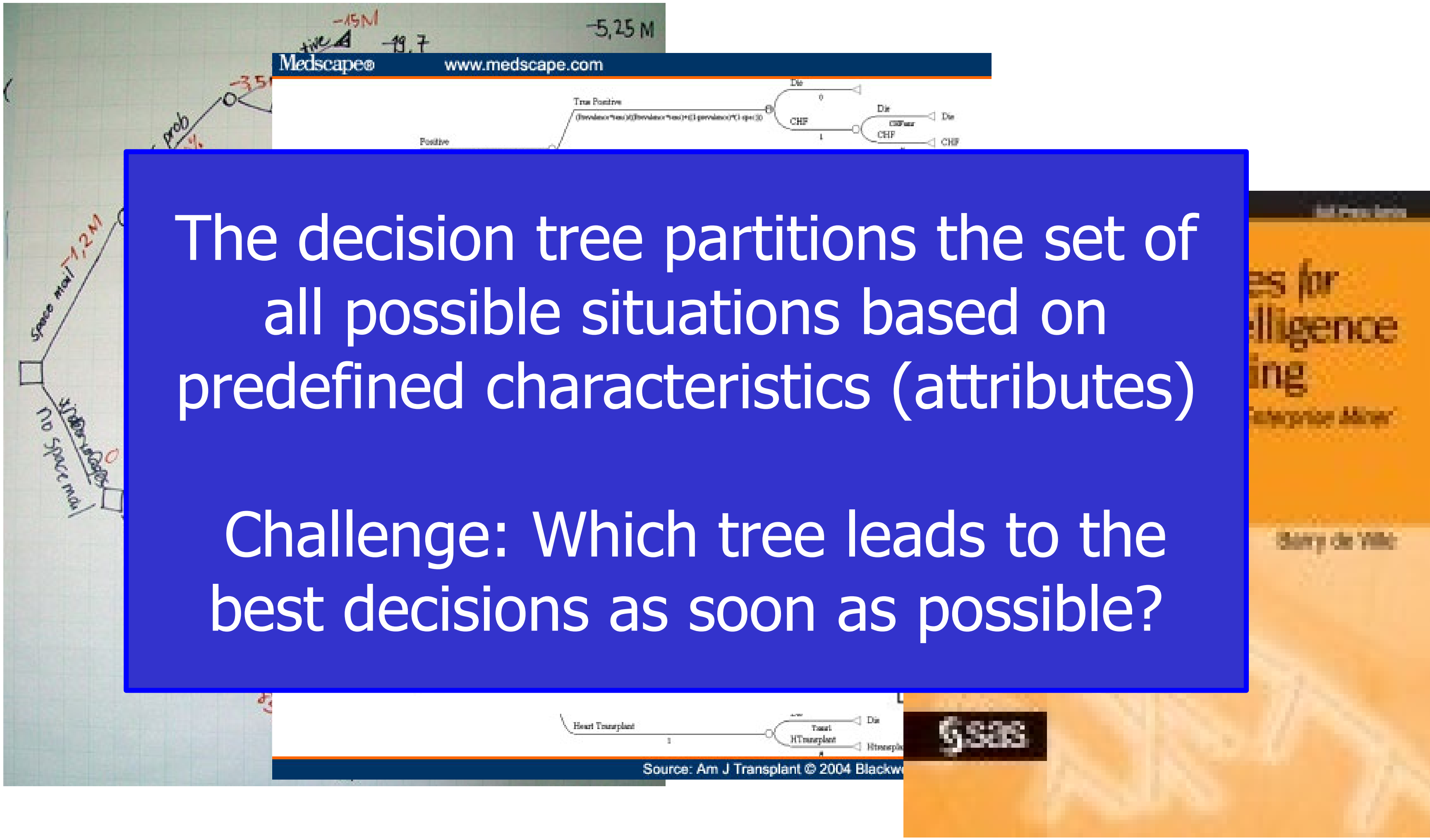
Outlook	Temperature	Humidity	Windy	Play
Sunny	Hot	High	False	No
Sunny	Hot	High	True	No
Overcast	Hot	High	False	Yes
Rainy	Mild	Normal	False	Yes
…	…	…	…	…

Decision Trees



Outlook	Temperature	Humidity	Windy	Play
Sunny	Hot	High	False	No
Sunny	Hot	High	True	No
Overcast	Hot	High	False	Yes
Rainy	Mild	Normal	False	Yes
…	…	…	…	…

Many Applications



The decision tree partitions the set of all possible situations based on predefined characteristics (attributes)

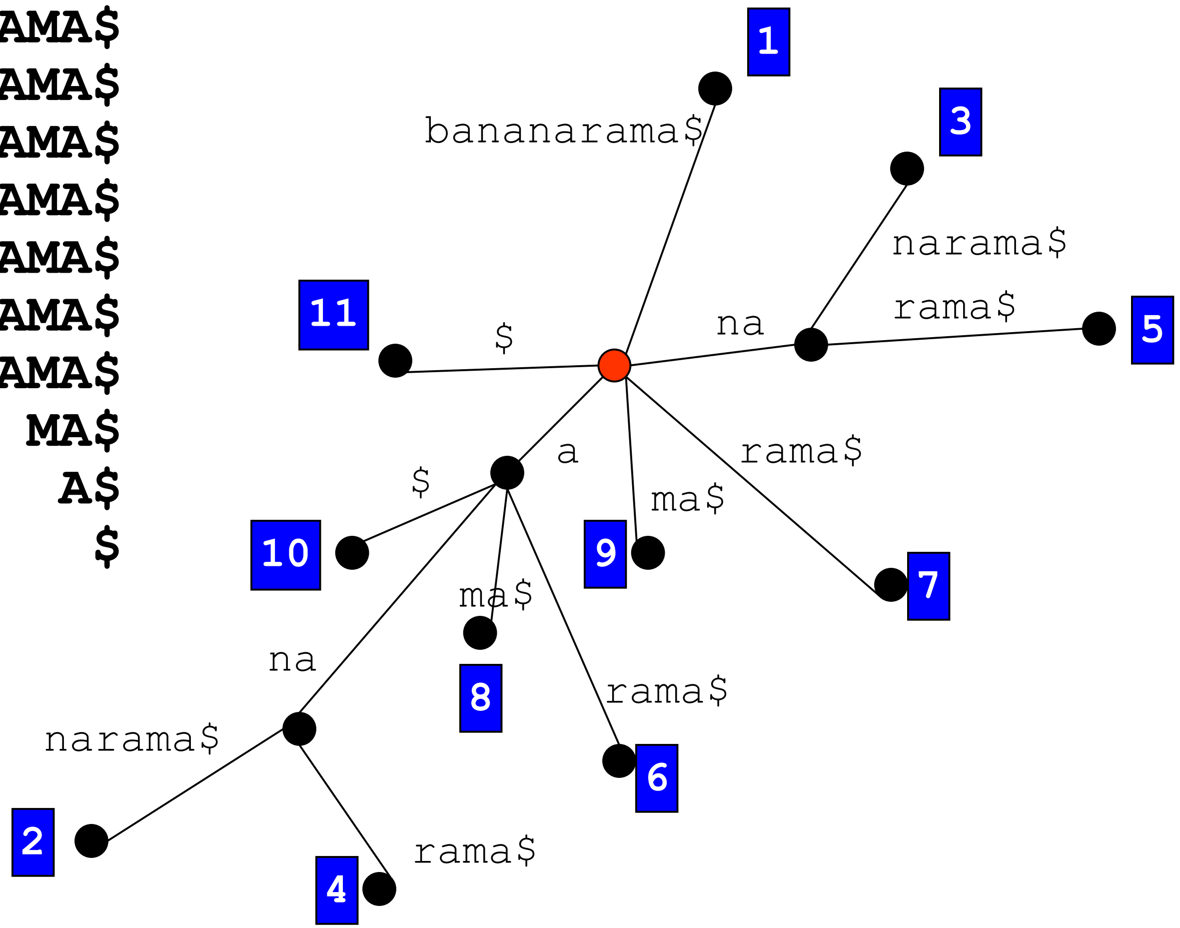
Challenge: Which tree leads to the best decisions as soon as possible?

Suffix-Trees

- Recall the problem to find all occurrences of a (short) string P in a (long) string T
- Fastest way (O(|P|)): Suffix Trees
  - Loot at all suffixes of T (there are |T| many)
  - Construct a tree
    - Every edge is labeled with a letter from T
    - All edges emitting from a node are labeled differently
    - Every path from root to a leaf is uniquely labeled
    - All suffixes of T are represented as leaves
- Every occurrence of P must be the prefix of a suffix of T
- Thus, every occurrence of P must map to a path starting at the root of the suffix tree

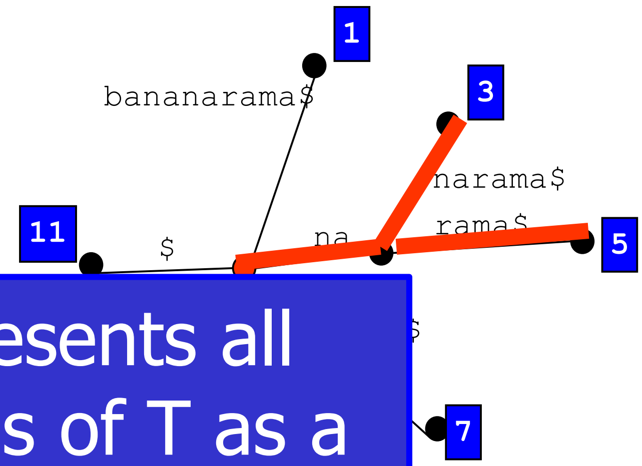
Example

```
12345678901
BANANARAMA$
 ANANARAMA$
  NANARAMA$
   ANARAMA$
    NARAMA$
     ARAMA$
      RAMA$
       AMA$
        MA$
         A$
          $
```



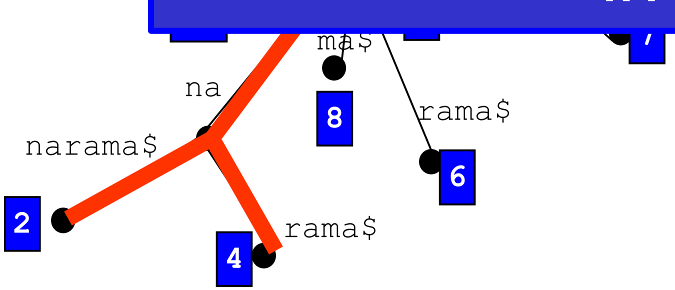
Searching in the Suffix Tree

P = „na“



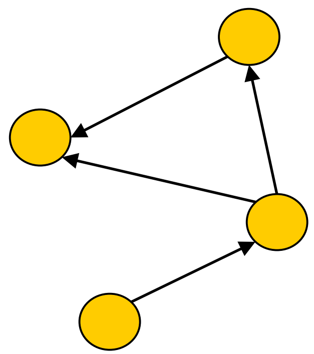
The suffix tree for T represents all common prefixes of suffixes of T as a unique path from root.

Challenge: Construction of a suffix tree in linear time.

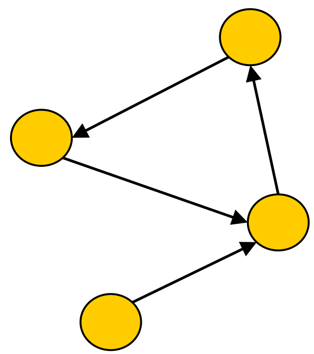


P = „an“

Not Trees

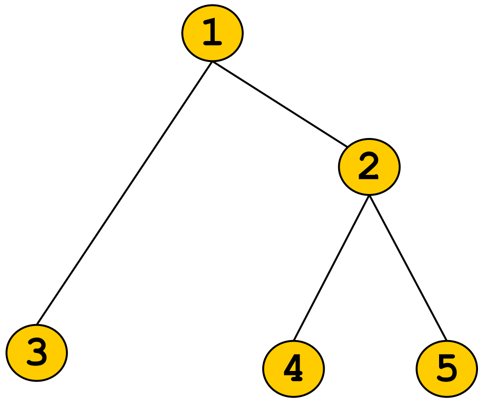


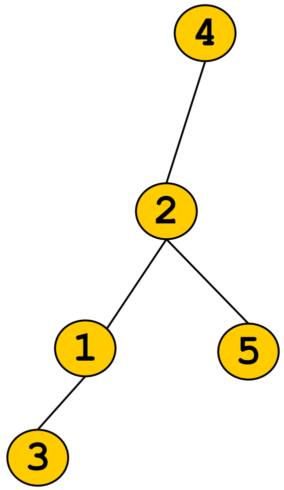
DAG: Directed,
acyclic graph



General
(directed) graph)

Directed? Single-rooted?





We sometimes draw undirected edges with root at the top and assume directed edges from root to leaves

Root: Only node without incoming edge

This visual aid is necessary! Otherwise, roots/leaves are not defined without directed edges

Graphs

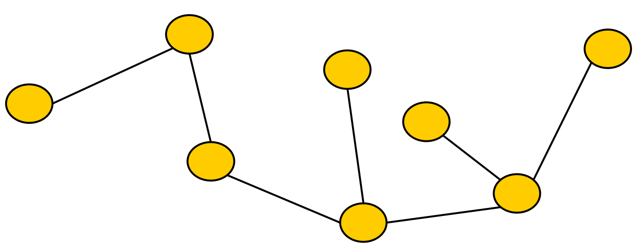
- Definition

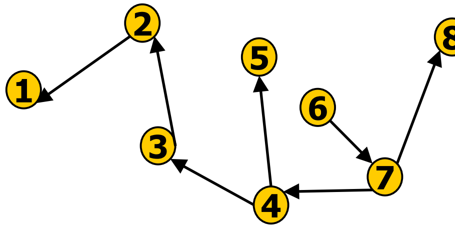
  A graph $G=(V, E)$ consists of a set V of vertices (nodes) and a set E of edges ($E \subseteq VxV$).
  - *A sequence of edges $e_1, e_2, .., e_n$ is called a path iff $\forall 1 \leq i < n-1$: $e_i=(v', v)$ and $e_{i+1}=(v, v'')$*
  - *The length of a path $e_1, e_2, .., e_n$ is n*
  - *A path $(v_1,v_2), (v_2,v_3), \ldots, (v_{n-1},v_n)$ is acyclic iff all v_i are different*
  - *G is connected if every pair v_i, v_j is connected by at least one path*
  - *G is undirected, if $\forall(v,v')\in E \Rightarrow (v',v)\in E$. Otherwise G is directed*
  - *G is acyclic if it contains no cyclic path*

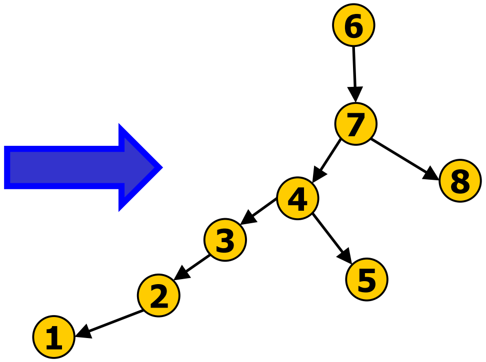
  Let $G=(V, E)$ be a directed graph and let $v,v'\in V$.
  - *Every edge $(v,v')\in E$ is called outgoing for v*
  - *Every edge $(v',v)\in E$ is called incoming for v*

Trees as Connected Graphs

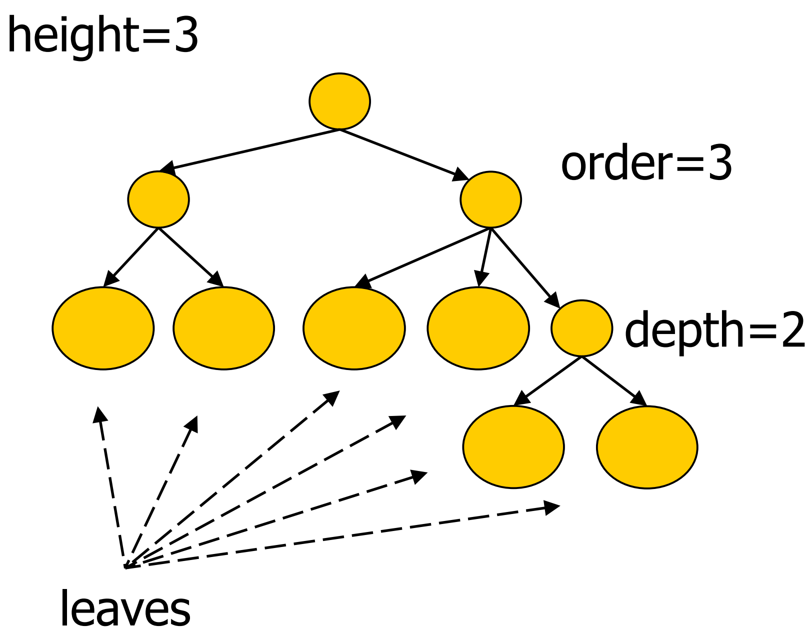
- Definition
  - *A undirected connected acyclic graph is called a undirected tree*
  - *A directed connected acyclic graph in which all but one vertex of in-degree 1 and one vertex has in-degree 0 is called a directed rooted tree*
- From now on: "Tree" means "rooted directed tree"
- Lemma
  - *In a tree, there exists exactly one path between root and any other node*





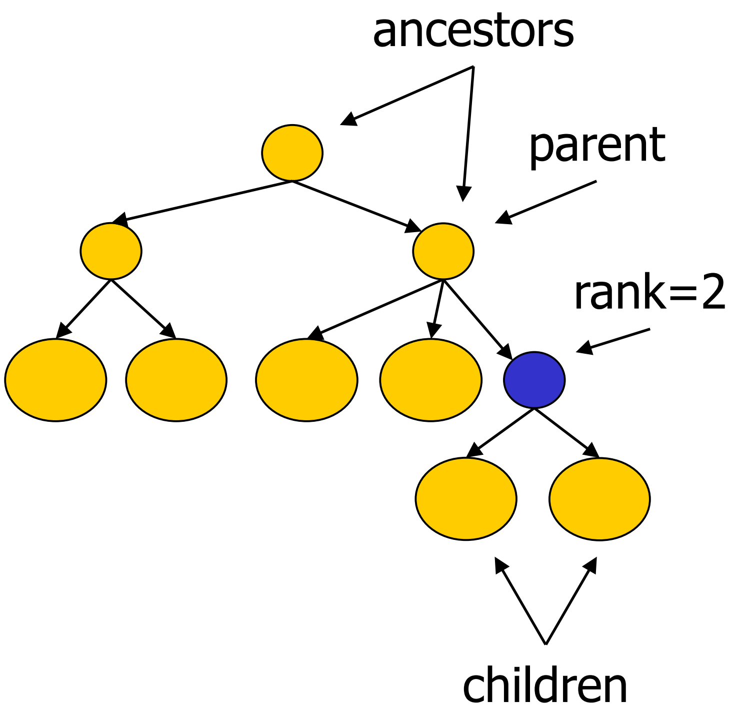


Terminology



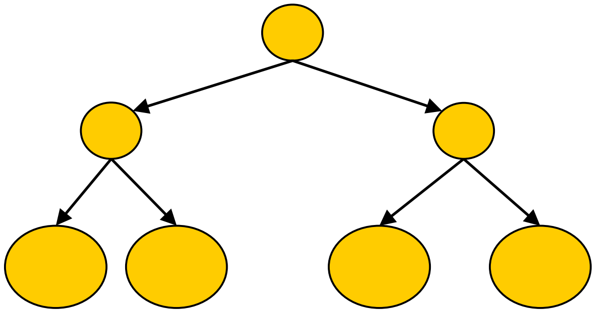
- Definition
  Let T be a tree. Then …
  - *A node with no outgoing edge is a leaf; other nodes are inner nodes*
  - *The depth of a node p is the length of the path from root to p*
  - *The height of T is the depth of its deepest leaf*
  - *The order of T is the maximal number of children of its nodes*
  - *"Level i" are all nodes at depth i*
  - *T is ordered if the children of all inner nodes are ordered*

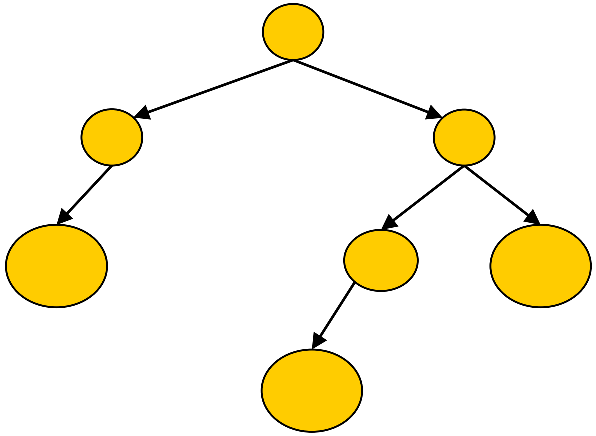
More Terminology



- Definition
  Let T be a tree and v a node.
  - *All nodes adjacent to an outgoing edge of v are v's children*
  - *v is called the parent of all its children*
  - *All nodes on the path from root to v without v are the ancestors of v*
  - *All nodes reachable from v are its successors*
  - *The rank of a node v is the number of its children*

Two More Concepts





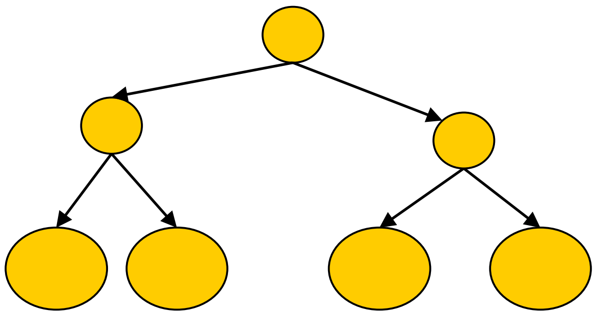
- Definition
  Let T be a directed tree of order k. T is complete if all its inner nodes have rank k and all leaves have the same depth

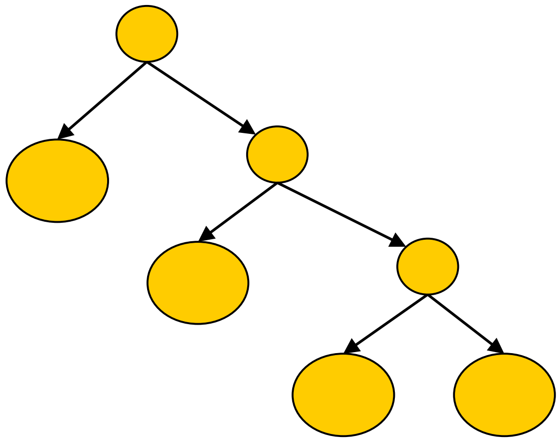
- In this lecture, we will mostly consider rooted ordered trees of order two (binary trees)

Recursive Definition of Trees

- Will often traverse trees using recursive functions
- Definition
  A (binary) tree is a structure defined as follows:
  - *A single node is a tree with height 0*
  - *If T_1 and T_2 are trees, then the structure formed by a new node v and edges from v to the root of T_1 and from v to the root of T_2 is a tree*
    - *v is its root*
    - *The height of this tree is max(height(T_1), height(T_2))+1;*
  - *If T_1 is a tree, then the structure formed by a new node v and an edge from v to the root of T_1 is a tree*
    - *v is its root*
    - *The height of this tree is height(T_1)+1;*

Some Properties (without proofs)





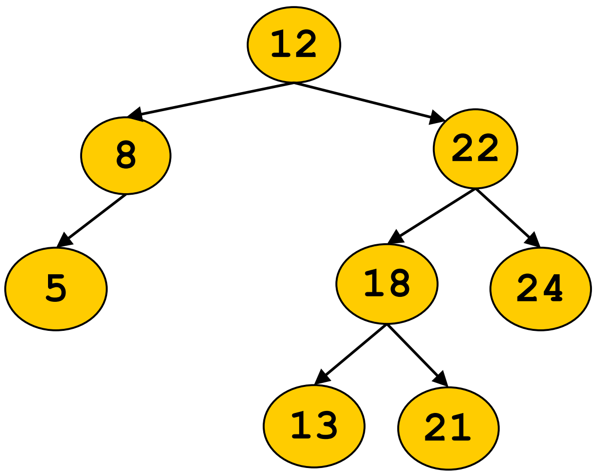
- Lemma
  Let T=(V, E) be a tree of order k. Then
  - *|V|=|E|+1*
  - *If T is complete, T has $k^{height(T)}$ leaves*
  - *If T is a complete binary tree, T has $2^{height(T)+1}-1$ nodes*
  - *If T is a binary tree with n leaves, height(T) ∈ [floor(log(n)), n-1]*

Content of this Lecture

- Trees
- Search Trees
  - Definition
  - Searching
  - Inserting
  - Deleting
- Natural Trees

Search Trees

- Definition
  A search tree $T=(V,E)$ is a rooted binary tree with $n=|V|$ differently key-labeled nodes such that $\forall v \in V$:
  - *label(v)>max(label(left_child(v)), label(successors(left_child(v)))*
  - *label(v)<min(label(right_child(v)), label(successors(right_child(v)))*
- Remarks
  - For simplicity, we use integer labels
  - "node" ~ "label of a node"
  - We only consider search trees without duplicate keys (easy to change)
  - Search trees are used to manage and search a list of keys
  - Operations: search, insert, delete



Search Trees

- Definition

  A *search tree* *T=(V,E) for a set of n unique keys is a labeled binary tree with |V|=n and*
  - *label(v)>max(label(left_child(v)), label(successors(left_child(v)))*
  - *label(v)<min(label(right_child(v)), label(successors(right_child(v)))*
- Remarks
  - For simplicity, we use integer labels
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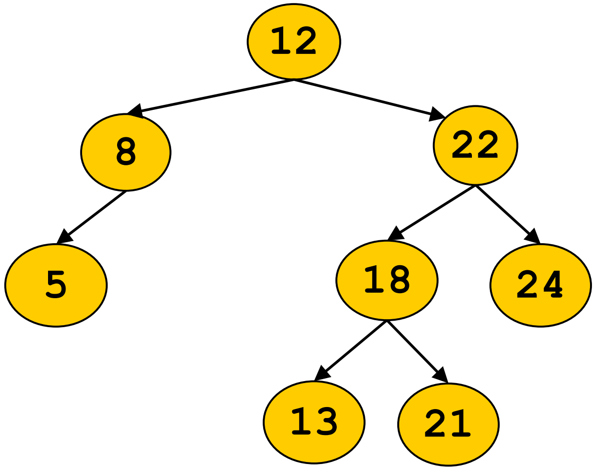
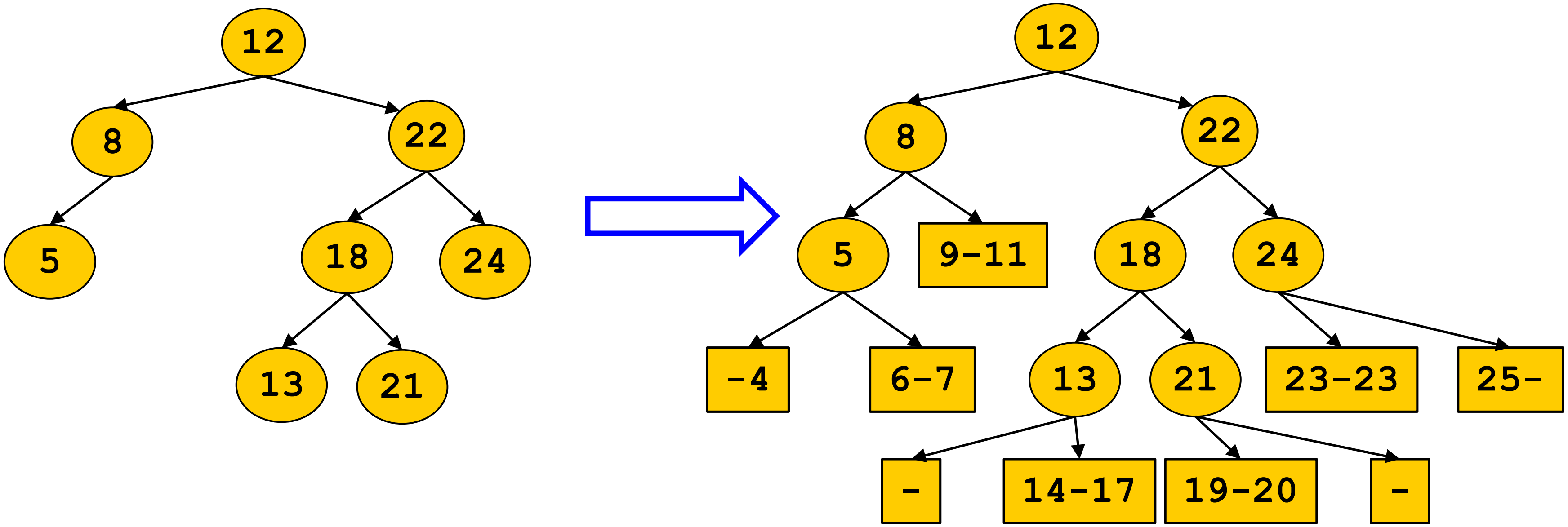


Complete Trees

- Conceptually, we pad search trees to full rank in all nodes
  - “padded” leaves are usually neither drawn nor implemented (NULL)
- A “padded” leaf represents the interval of values that would be below this node



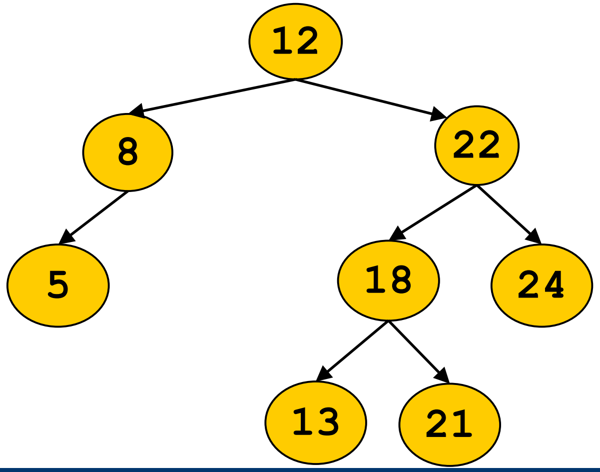
What For?

- For a search tree T=(V,E), we eventually will reach O(log(|V|)) for testing whether k∈T and for inserting and deleting a key
  - First: Average Case of natural trees
  - Next: Worst Case for AVL-Trees
- Compared to binsearch on arrays, search trees are a dynamically growing / shrinking data structure
  - But need to store pointers
  - Complete trees can be easily managed in arrays

Searching

- Searching a key k
  - Comparing k to a node determines whether we have to look further down the left or the right subtree
    - We stop if label(node)=k
  - If there is no child left, k∉T
- Complexity
  - In the worst case we need to traverse the longest path in T to show k∉T
  - Thus: O(|V|)
  - Wait a bit …

```
func node search( T search_tree,
                  k integer) {
  v := root(T);
  while v!=null do
    if label(v)>k then
      v := v.left_child();
    else if label(v)<k then
      v := v.right_child();
    else
      return v;
  end while;
  return null;
}
```

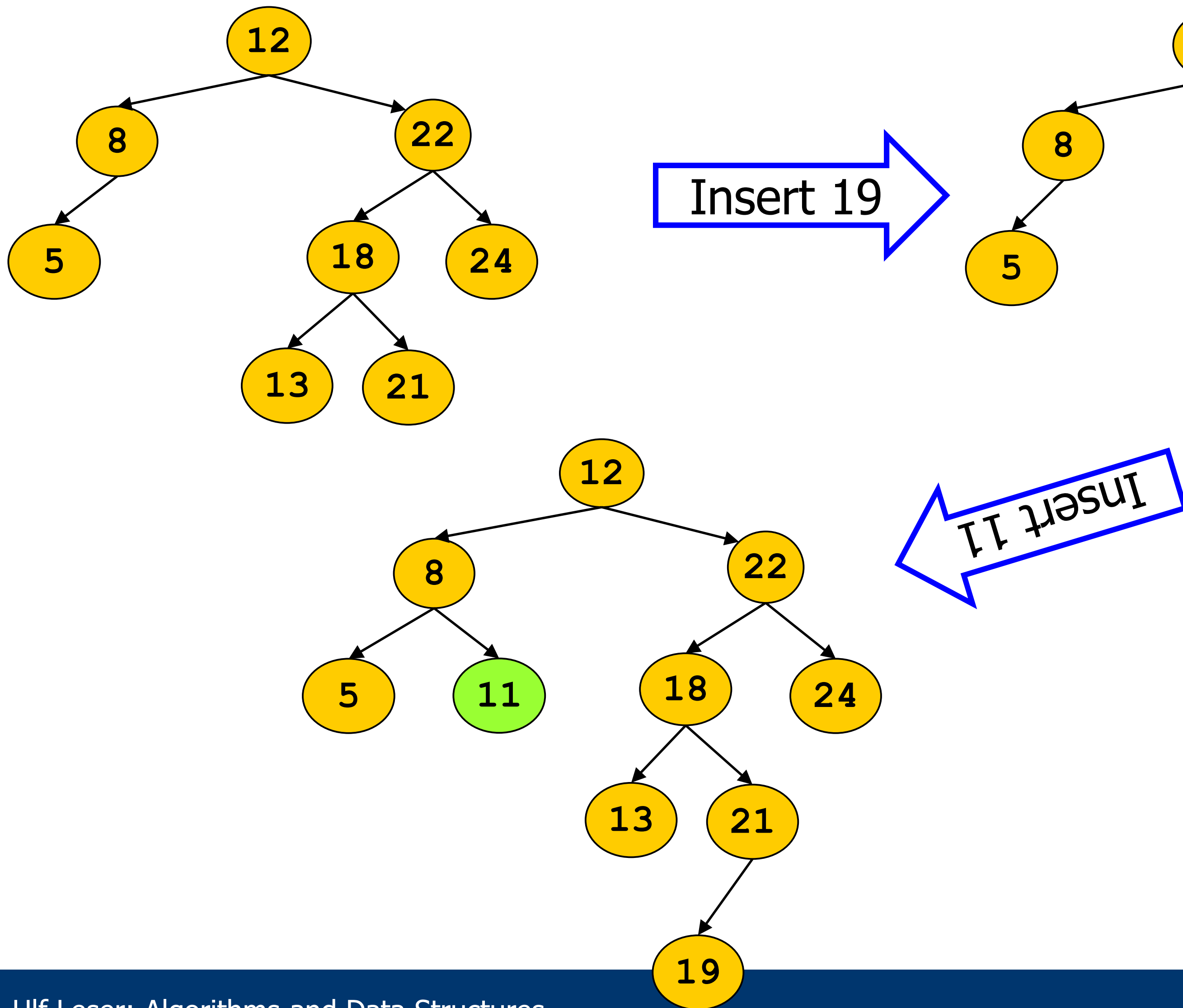


Insertion

```
func bool insert( T search_tree,
                  k integer) {
  v := root(T);
  while v!=null do
    p := v;
    if label(v)>k then
      v := v.left_child();
    else if label(v)<k then
      v := v.right_child();
    else
      return false;
  end while;
  if label(p)>k then
    p.left_child := new node(k);
  else
    p.right_child := new node(k);
  end if;
  return true;
}
```

- First search the new key k
  - If k∈T, we do nothing
  - If k∉T, the search must finish at a null pointer in a node p
    - A "right pointer" if label(p)<k, otherwise a "left pointer"
- We replace the null with a pointer to a new node k
- Complexity: Same as search

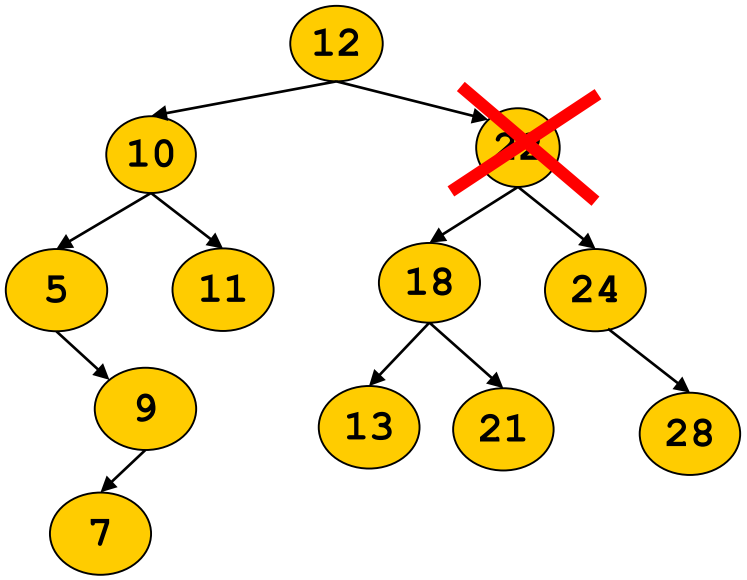
Example



Deletion

- Again, we first search k
- If $k \notin T$, we are done
- Assume $k \in T$. The following situations are possible
  - k is stored in a leaf. Then simply remove this leaf
  - k is stored in an inner node q with only one child. Then remove q and connect parent(q) to child(q)
  - k is stored in an inner node q with two children. Then …

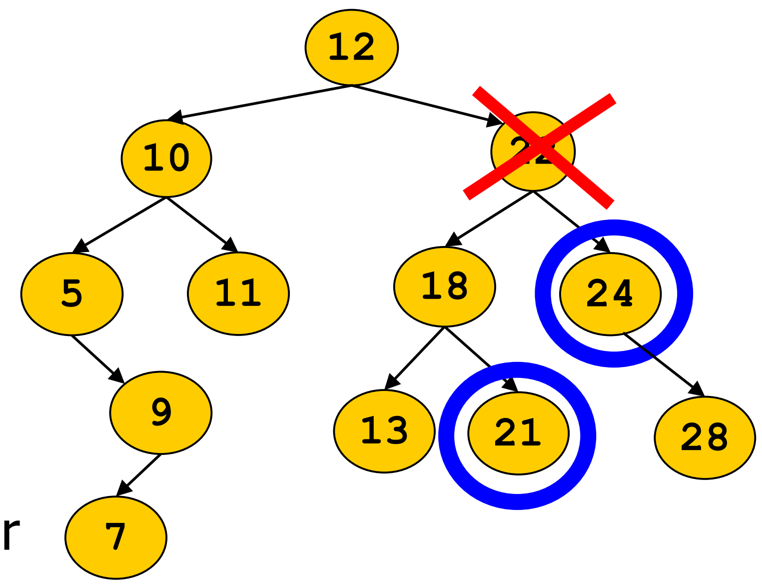
Observations



- We cannot remove q, but we can replace the label of q with another label - and remove this node
- We need a node q' which can be removed and whose label k' can replace k without hurting the search tree constraints
  - label(k')>max(label(left_child(k')), label(successors(left_child(k')))
  - label(k')<min(label(right_child(k')), label(successors(right_child(k')))

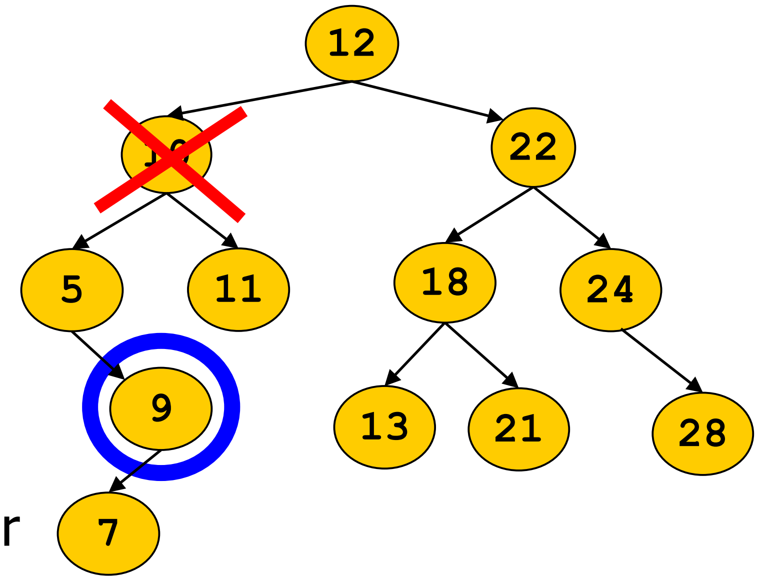
Observations

- Two candidates
  - Largest value in the left subtree (symmetric predecessor of k)
  - Smallest value in the right subtree (symmetric successor of k)
- We can choose any of those
  - Let’s use the symmetric predecessor
  - This is either a leaf – no problem

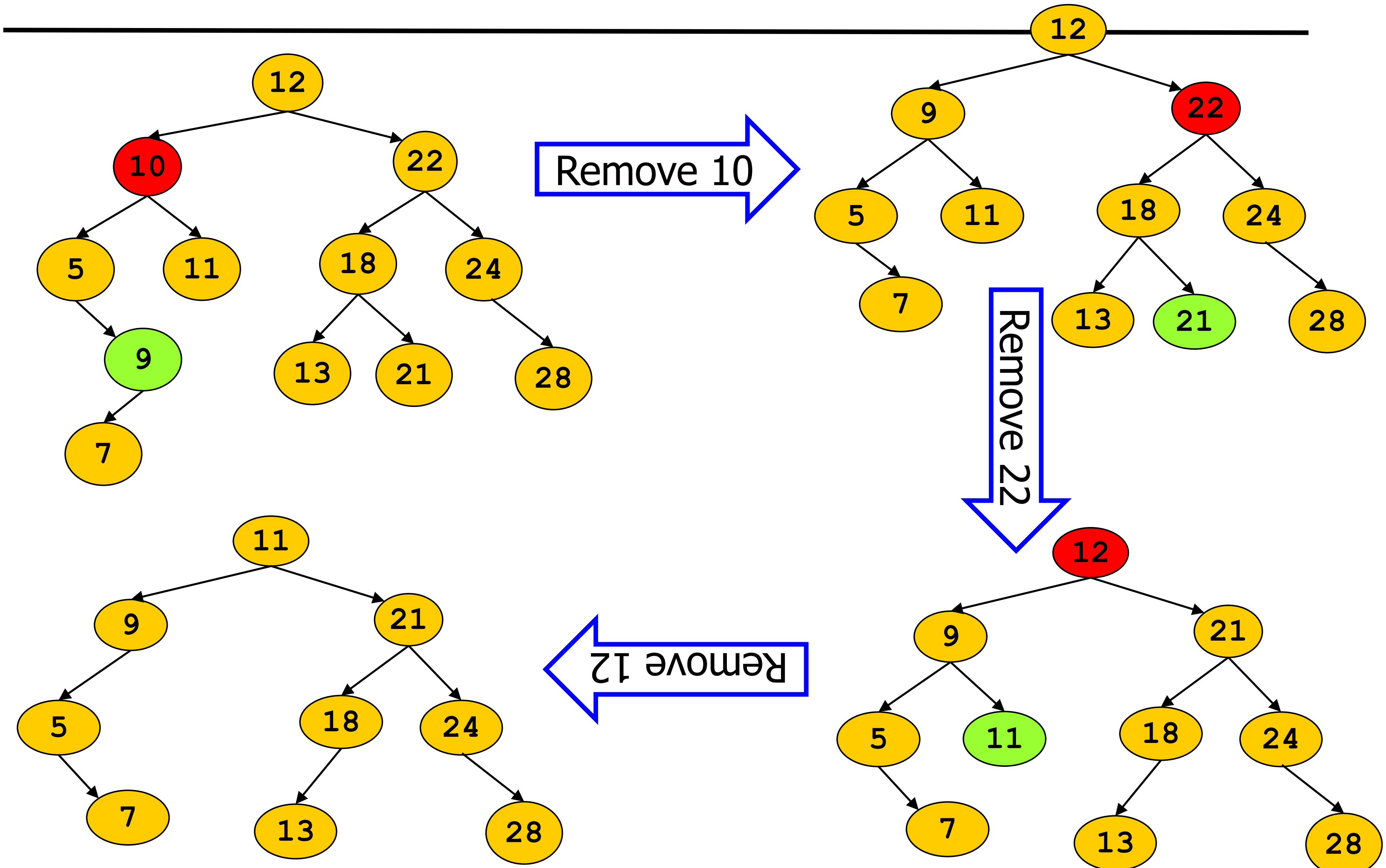


Observations

- Two candidates
  - Largest value in the left subtree (symmetric predecessor of k)
  - Smallest value in the right subtree (symmetric successor of k)
- We can choose any of those
  - Let's use the symmetric predecessor
  - This is either a leaf
  - Or an inner node; but since its label is larger than that of all other labels in the left subtree of q, it can only have a left child
  - Thus it is a node with one child - and can be removed easily



Example



12
10
22
5
11
18
24
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7
Remove 10
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Remove 22
12
9
21
5
11
18
24
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Remove 12
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24
7
13
28

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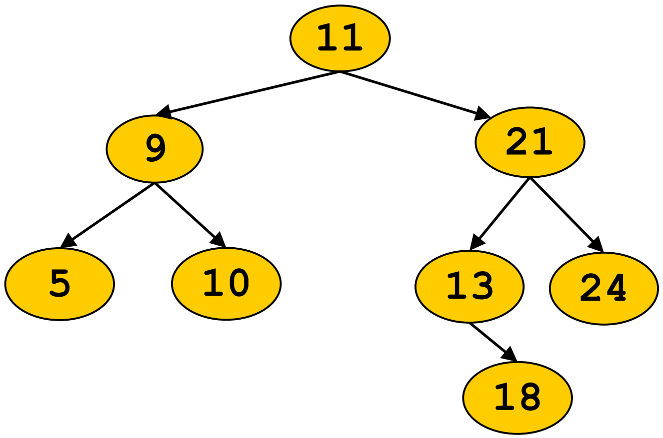
- Natural Trees

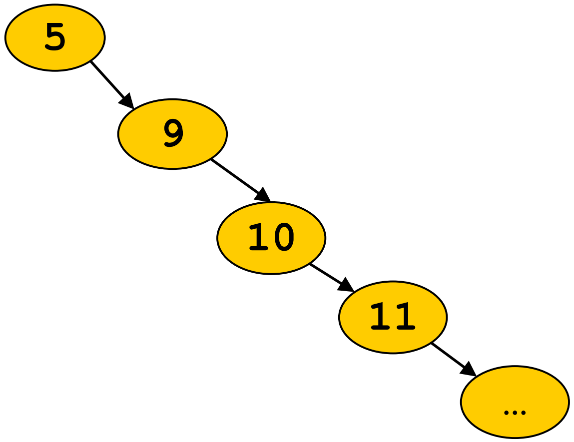
Natural Trees

- A search tree T created by inserting and deleting n keys in random order is called a natural tree
- As any binary tree, it has height(T)∈[n-1, log(n)]
- Height depends on the order in which keys were inserted
- Example

11,9,10,5,21,13,24,18



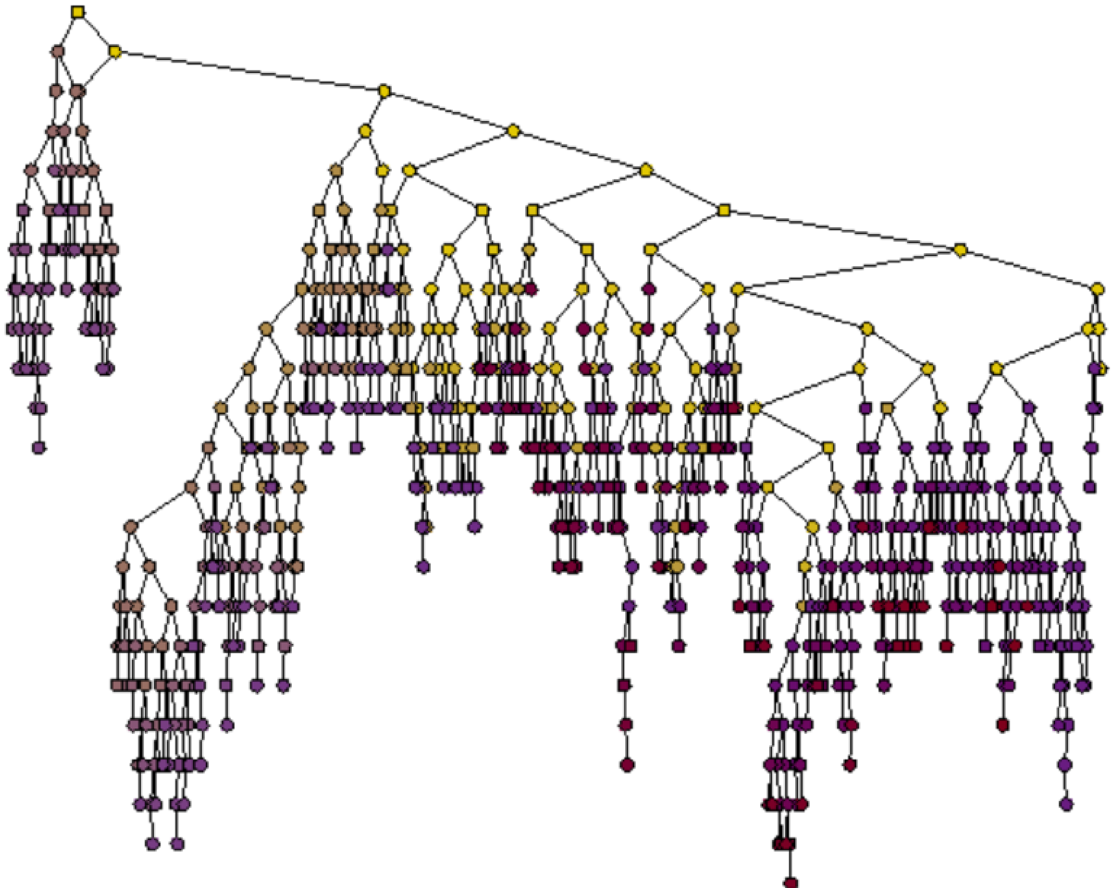
5,9,10,11,13,18,21,24



Average Case

- A natural tree with n nodes has maximal height of n-1
- Thus, searching will need O(n) comparisons in worst-case
  - Same for inserting and deleting
- But: Natural trees are not bad on average
  - The average case is O(log(n))
  - More precisely, a natural tree is on average only ~1.4 times deeper than the optimal search tree (with height h~log(n))
  - We skip the proof (argue over all possible orders of inserting n keys), because balanced search trees (AVL trees) are O(log(n)) also in worst-case and are not much harder to implement

Example



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Exemplary Questions

- Construct a natural search tree from the following input, showing all intermediate steps (I: insert; D: delete): I5, I7, I3, I10, D7, I7, I13, I12, D5
- The worst case complexity for inserting/deleting a key into a search tree with n=|V| nodes is O(n). Give an order of the following operations such that this worst case happens for every operation: I5, I7, I3, I10, D7, I7, I13, I12, D5
- For deleting a given key k in a natural search tree, one may need to find the symmetric predecessor (SP) of a key. Define what a SP is, give an algorithm for finding it (starting from k), and analyze its complexity