Algorithms and Data Structures

Minimal Spanning Trees

Ulf Leser

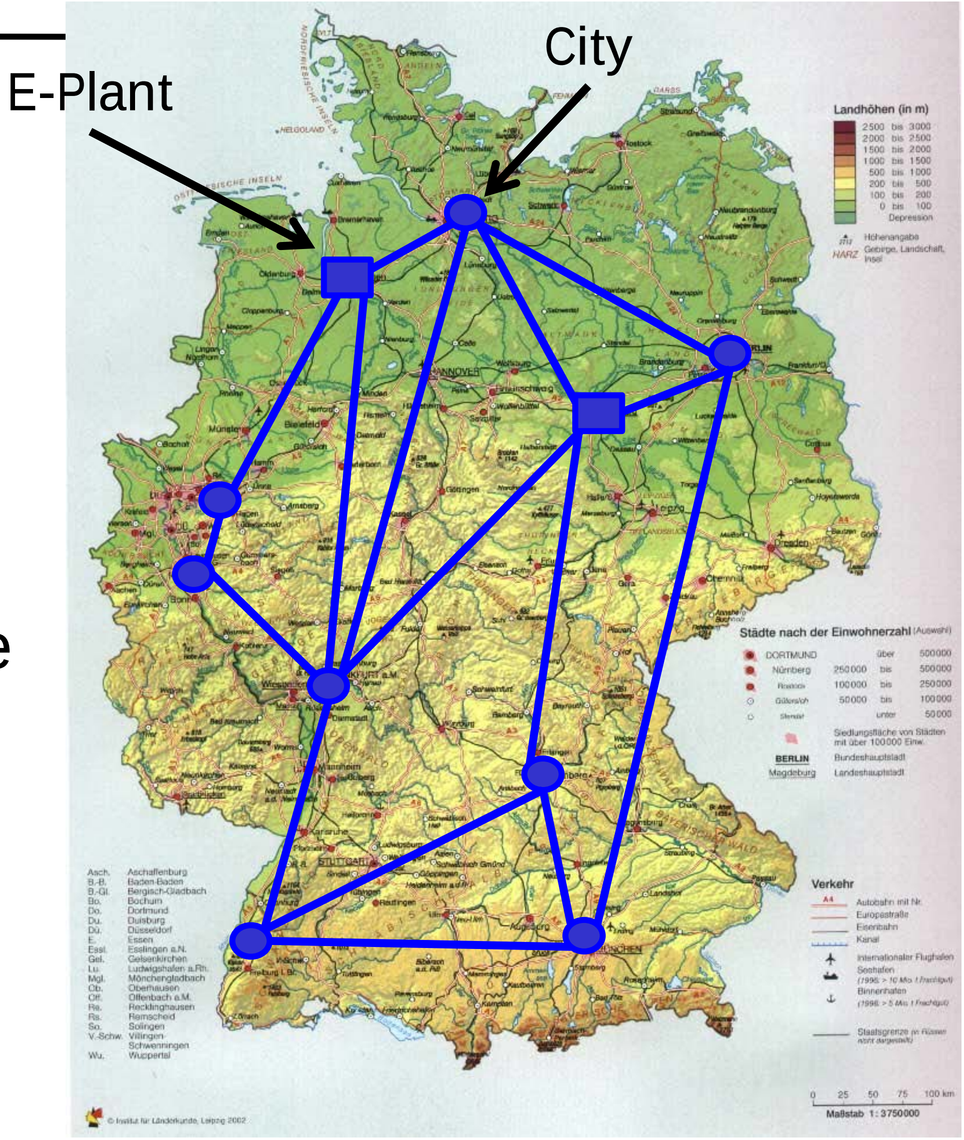
Die Energiewende

- Electricity is created in many more places than before
- Electricity is consumed in many places
- Places of production are not evenly distributed across the country
- We need to build new electricity highways

Source: http://www.deutsche-mittelgebirge.de/

Die Energiewende

- How can we do this as cheap as possible?
- Not all connections are possible
  - Mountains, rivers, ...
- Different connections have different costs

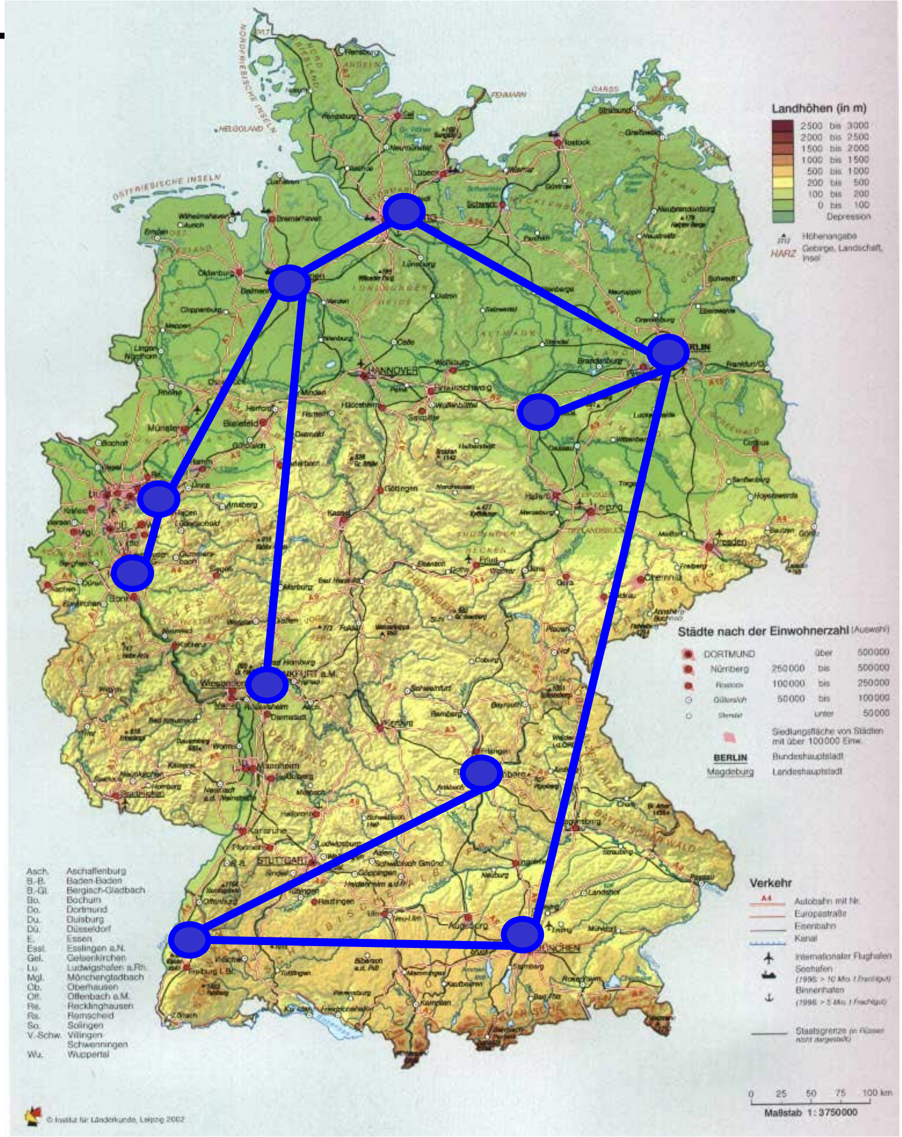


Die Energiewende

- Requirement for a solution: Every city and every plant must be connected to the network
  - We treat them uniformly
  - We don't care about the length of a connection
- One solution

Die Energiewende

- Another solution
- Of course, in real life we may build crossroads outside cities

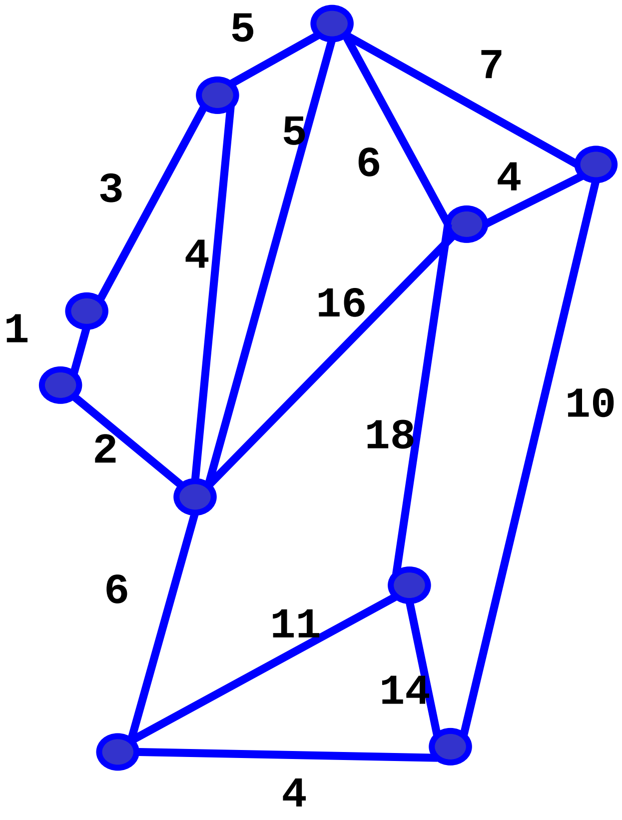


Die Energiewende

- This is the Steinerbaum-Problem
  - Some nodes must be connected, other nodes maybe connected
- Optimal solution is much harder to find
- Not considered here

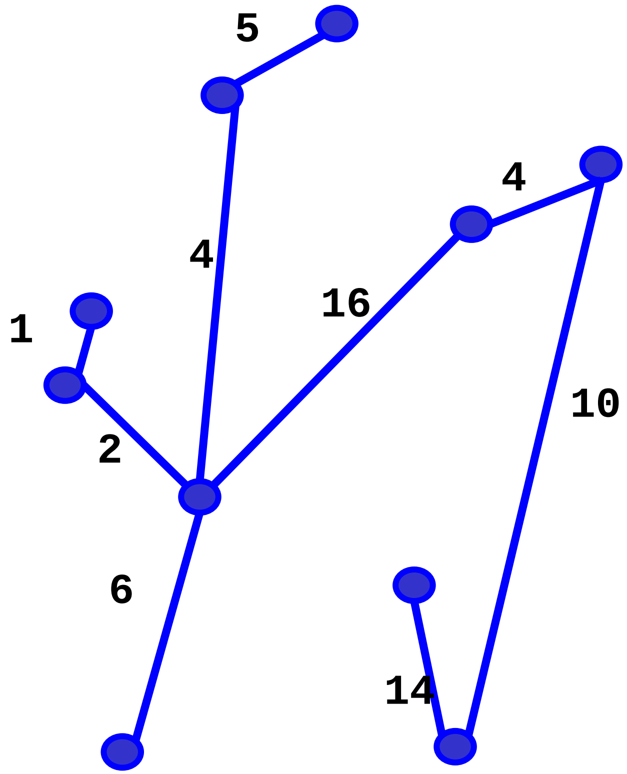
Abstraction

- Given an undirected, positively weighted, connected graph G=(V,E)
- Find a subset E'⊆E such that cost(E') is minimal and G'=(V, E') is connected
  - cost(E'): Sum of the edge weights
- Every such E' (or G') is called a minimum spanning tree (MST) for G



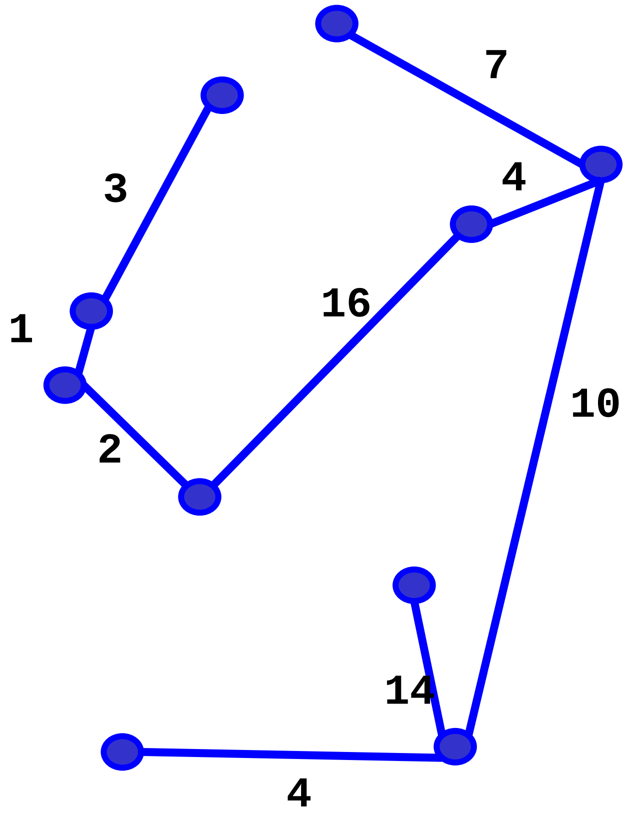
Example 1

- Cost = 62



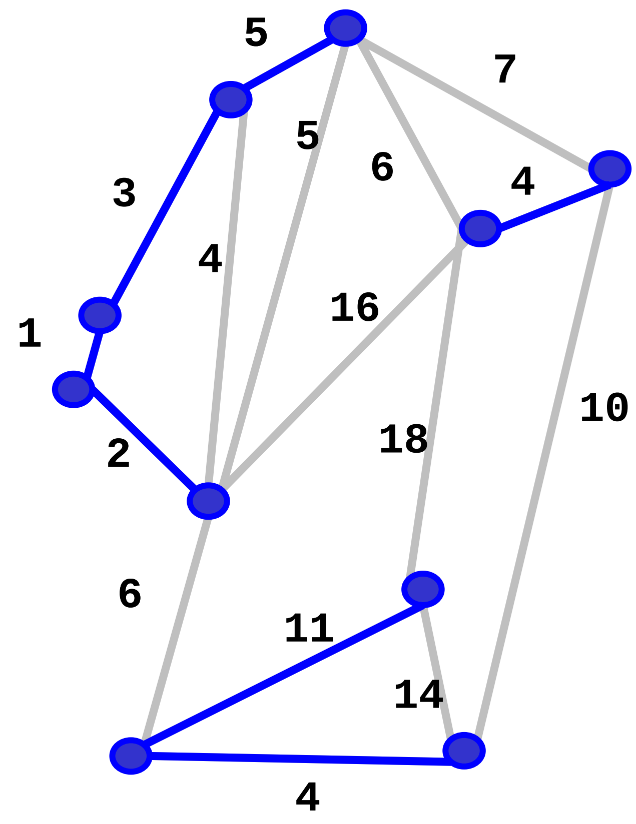
Example 2

- Cost = 61



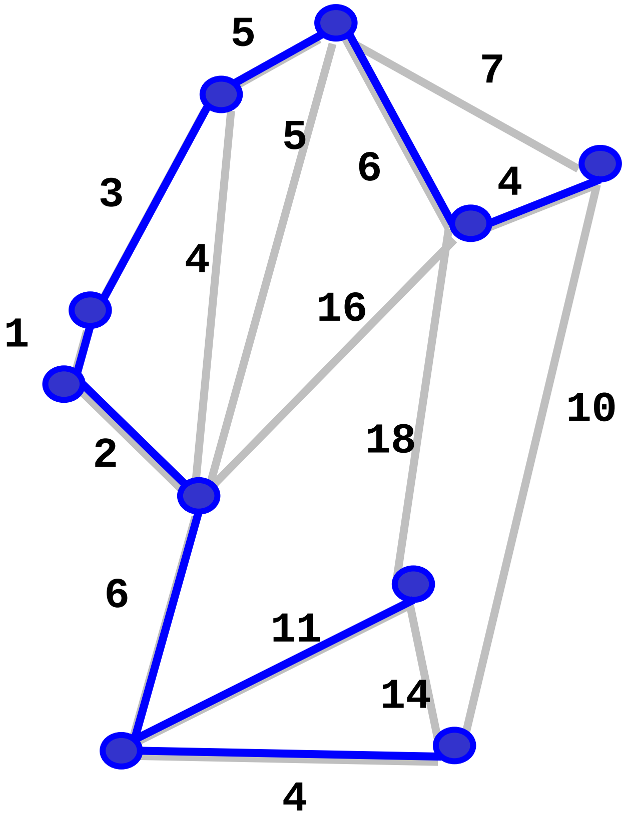
First Algorithm

- Let's try greedy
  - Sort edges by weight
  - Add the next cheapest edge to E' whenever it connects a new node to something already known
- Hmm



Second Algorithm

- Let's try greedy – another way
  - Sort edges by weight
  - Add cheapest edge to E'
  - Add all edges to E' in ascending order such that every new edge adds a new node to the graph induced by E'
  - Repeat until E' is complete
- Cost = 42
  - Is this optimal?
  - Does this always work?
  - How can we implement this algorithm efficiently?



Overview

- First algorithms for computing MST date back to the 1920s
- Algorithms are not difficult; much research went into efficient implementations
- Actually, MSTs can be computed in a greedy manner
- Algorithms need not grow only one component; in general, we may have “connected islands” that all get connected to one component in the end
- In each step, one needs to decide which edge to add next to which island (or which edges not to add)
- What are criteria for adding / not adding edges?

Content of this Lecture

- Minimal Spanning Trees
- Basic Properties
  - Tree
  - Cuts
  - Cycles
- Algorithms
- Implementation

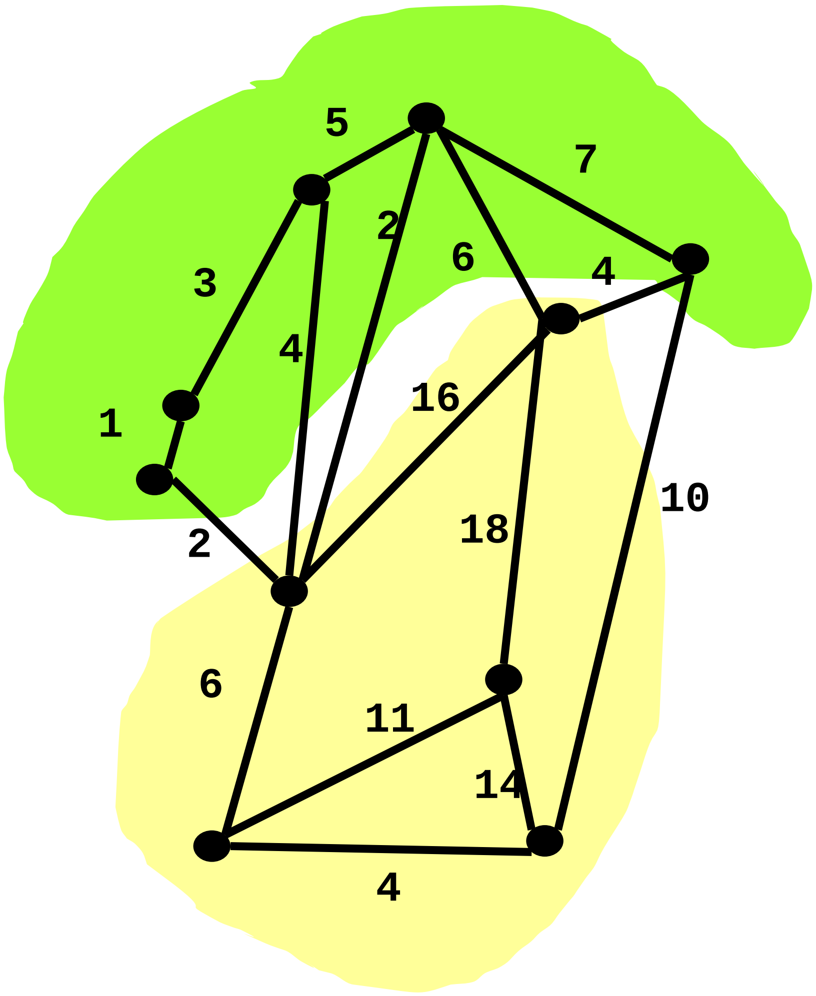
Minimal Spanning Trees

- Lemma
  Let $G=(V, E)$ and $E' \subseteq E$ be the subset of E' with minimal cost such that G', the graph induced by E', is connected. Then G' is a tree.
- Proof
  - Recall: A (undirected) tree is a undirected, connected acyclic graph
  - By definition, G' is connected and undirected
  - Imagine G' had a cycle. Then G' cannot have minimal cost, because removing any of the edges on the cycle from E' would create a subset E'' that has less cost, and the induced subgraph would still be connected
    - We assumed all edge weights to be positive
- Note: If all edge weights are distinct, the MST is unique

Cuts

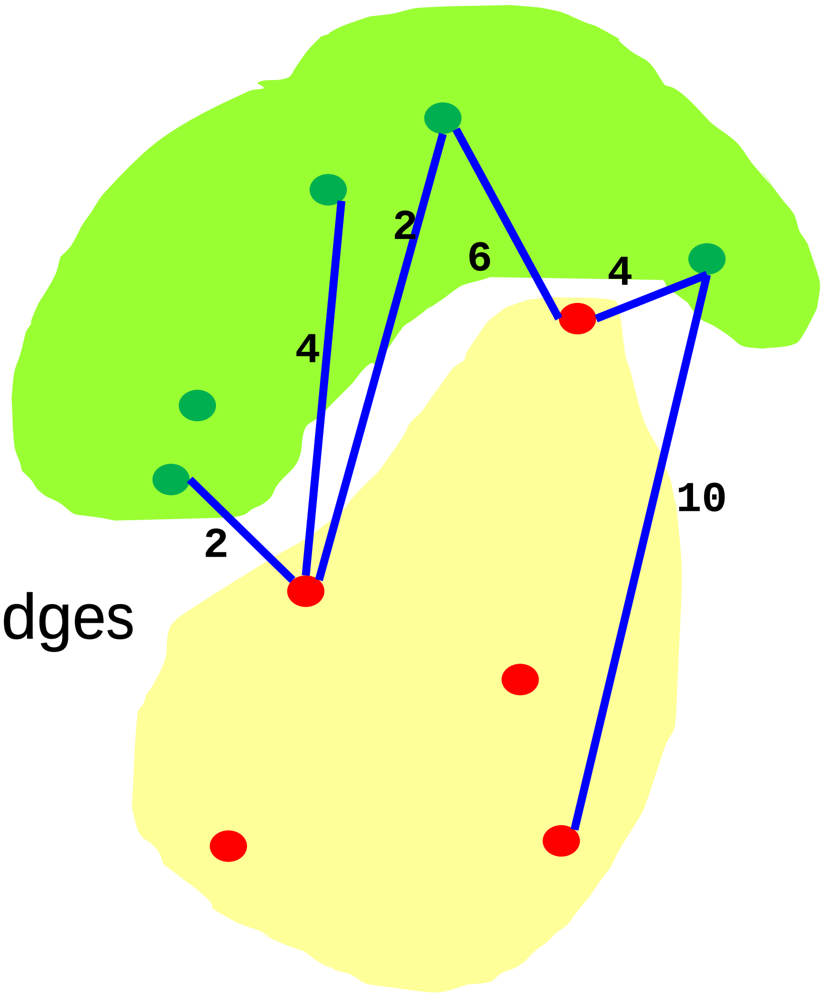
- Definition
  Let G=(V, E). A cut is a binary partitioning of V into two sets V_1, V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.
- Lemma
  Let G=(V, E) and V_1, V_2 be a cut of V. Let F be the set of all edges going from any node in V_1 to any node in V_2. Let F' be those edges of F with minimal weight. Then any MST G' of G must contains one edge of F', and every edge of F' is contained in at least one MST of G
- Remarks
  - This holds for arbitrary cuts – a very powerful statement
  - Edges in F are called crossing edges

Example



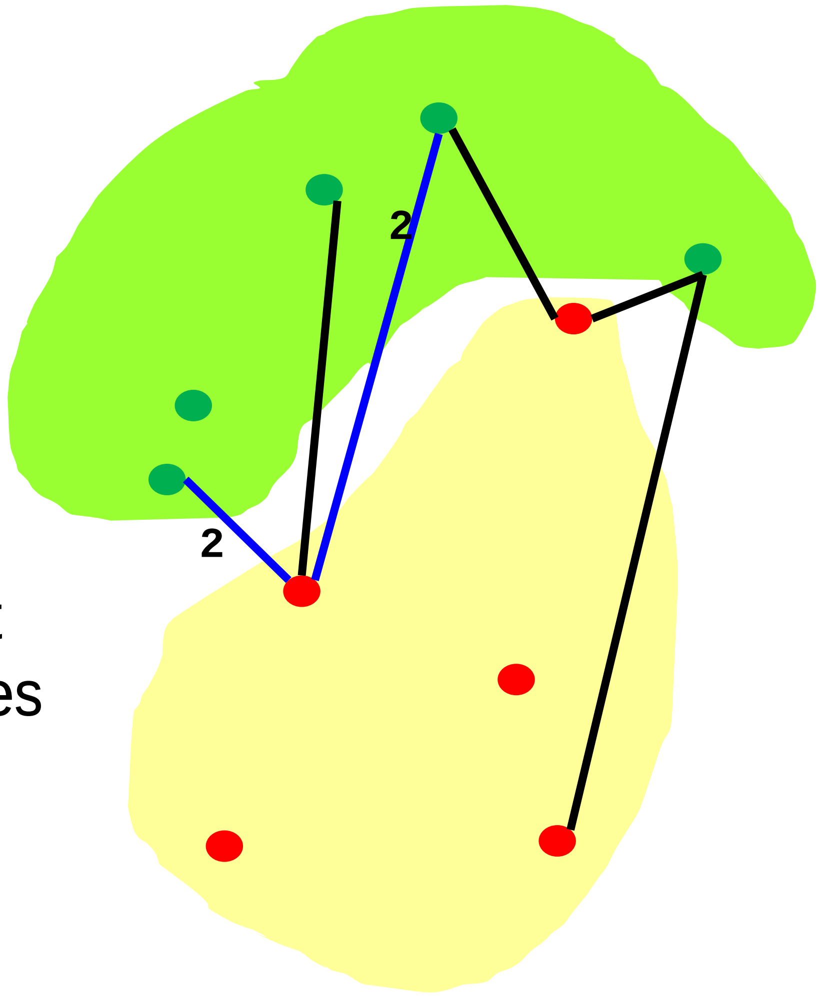
Example

- F:
  All crossing edges



Example

- F':
  The cheapest crossing edges

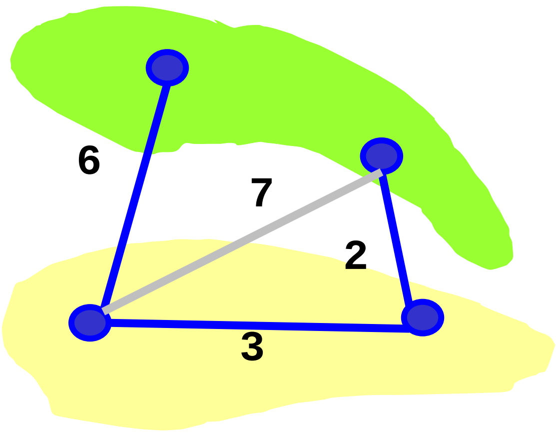


Proof

- Every MST G' contains one $f \in F'$
  - Imagine a G' that has no such f. Still, G' must be connected, so it must contain at least one of the crossing edges from F. Assume it contains only one such edge, h. h must have a higher weight than f because $h \notin F'$. Further, V_1 and V_2 must be connected in themselves. Then G' cannot be minimal, because removing h and adding some $f \in F$ would create a cheaper MST – contradiction.
  - Same argument holds if G' contains more than one crossing edge, all of which are not minimal
- Every $f \in F'$ is contained in at least one MST
  - Imagine f is not contained in any MST. Let G' be a MST and h be the edge in G' connecting V_1 and V_2. h must be in F', or G' is not minimal. Thus, the MST formed by removing h and adding f also is a MST – contradiction.

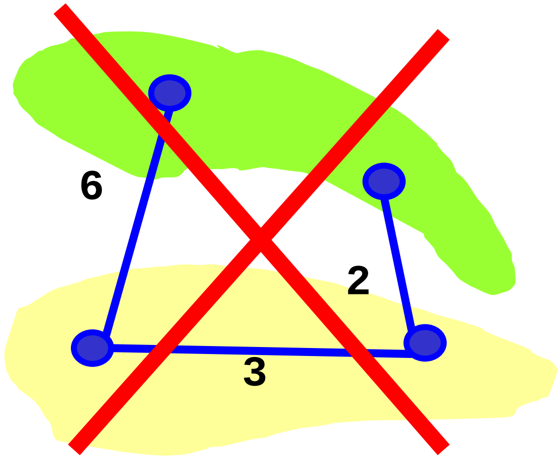
Beware

- For a given cut V_1, V_2, a MST G‘ may contain more than one crossing edge (and at least one must have minimal weight)



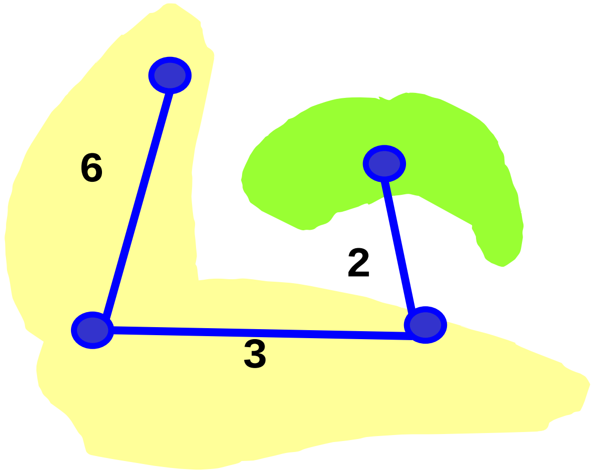
Consequences

- The cut property is a powerful tool for computing MSTs
- Lemma (cut property)
  Let G=(V, E) and G'=(V, E') be a MST of G. Then every $e \in E'$ has minimal cost among all crossing edges of the cut V_1, V_2 formed by removing e from G'.
- Proof
  - Since G' is a tree, every edge from E' "cuts" G
  - Rest follows from previous lemma
- Can be used to check whether a given E' is a MST



Consequences

- The cut property is a strong help for computing MSTs
- Lemma (cut property)
  Let G=(V, E) and G‘=(V, E‘) be a MST of G. Then every $e \in E'$ has minimal cost among all crossing edges of the cut V_1, V_2 formed by removing e from G‘.
- Proof
  - Since G’ is a tree, every edge from E‘ “cuts” G
  - Rest follows from previous lemma
- Can be used to check whether a given E’ is a MST



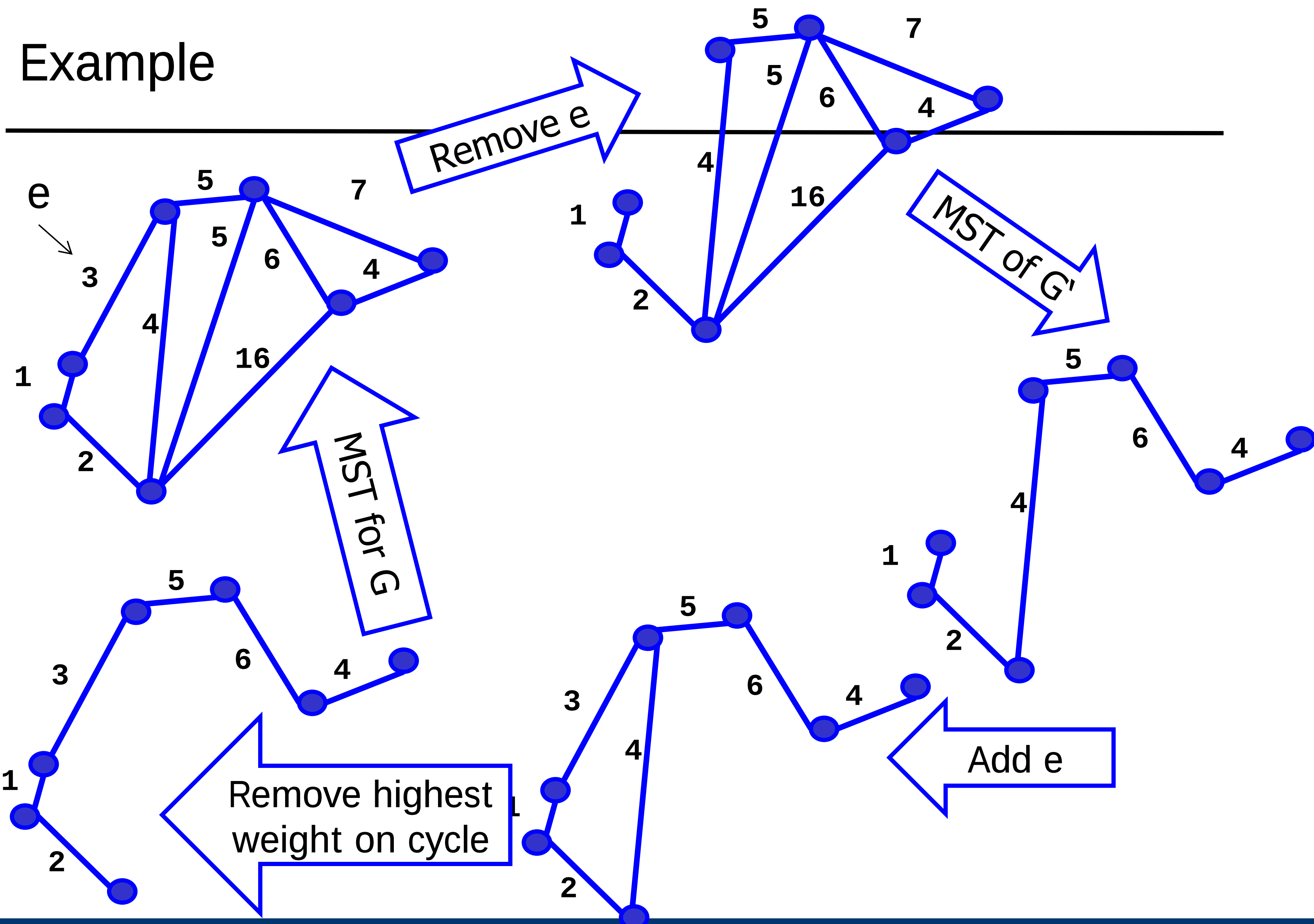
Content of this Lecture

- Minimal Spanning Trees
- Basic Properties
  - Tree
  - Cuts
  - Cycles
- Algorithms
- Implementation

Cycles

- Lemma (cycle property)
  Let G=(V, E) and G'=(V, E') with E'=E\e for some edge e such that G' still is connected. Let T' be a MST for G'. When we add e to T' and remove the edge with the highest weight on the then introduced cycle in T', forming T, then T is a MST for G.
- Proof idea
  - Adding e to T' must build a cycle because T' is a MST over V
  - Removing any of the edges on the cycle still leaves a connected tree
  - Removing the most expensive one leaves the minimal tree

Example


e
Remove e
MST of G'
MST for G
Add e
Remove highest weight on cycle

Implications

- T’ is a MST for G without e
- Imagine we would enumerate edges in some order
- Taking into account a new edge e may allow us to replace an edge in T’ with a cheaper one, creating a “better” MST for G
  - If e is not the edge with the highest weight on the cycle
- This means that an edge with maximal weight on a cycle in G cannot be part of any MST of G

Content of this Lecture

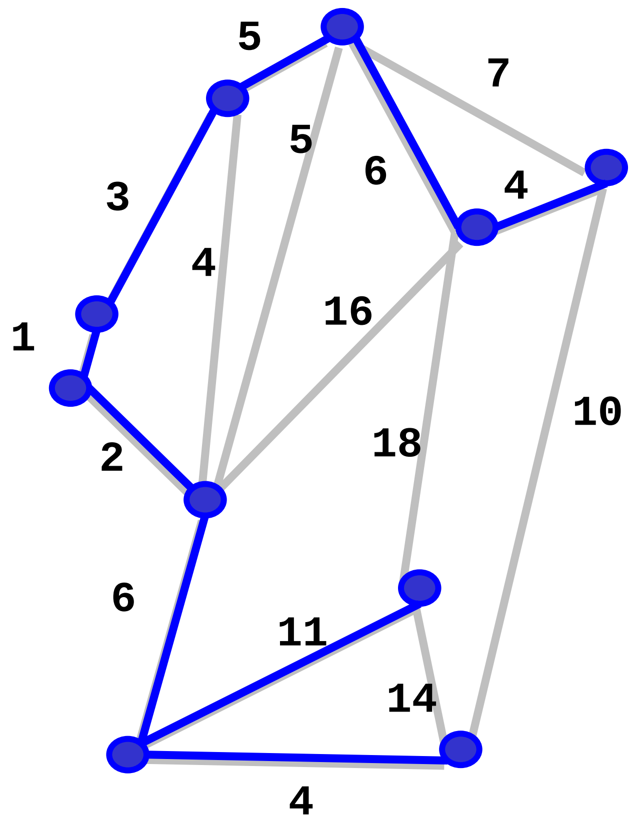
- Minimal Spanning Trees
- Basic Properties
- Algorithms
  - R.C. Prim: Shortest connection networks and some generalizations. Bell System Technical Journal, 1957
    - Also Jarnik, Prim, Dijkstra: Jarník, 1930 – Prim, 1957 – Dijkstra , 1959
  - J. Kruskal: On the shortest spanning subtree and the traveling salesman problem. Proc. of the American Mathematical Soc., 1956
  - Otakar Borůvka: O jistém problému minimálním (Über ein gewisses Minimierungsproblem), 1926
  - [Wikipedia, OW93]
- Implementation

Prim's Algorithm

Greedy; we never make mistakes

- Recall cut property: Every edge e in a MST is a minimal edge among the two partitions created by removing e
- Prim's Algorithm
  Start with an empty tree T. Continue adding the edge e with the lowest cost to T such that e connects T with a new node until all nodes of G are in T. Then T is a MST.
- Proof
  - Consider, at each stage, nodes in T as one partition V_1 and all other nodes as the other partition V_2
  - By cut property, the cheapest crossing-edge between V_1 and V_2 must be in the MST
  - Since we only add those edges, T finally must be a MST

Example

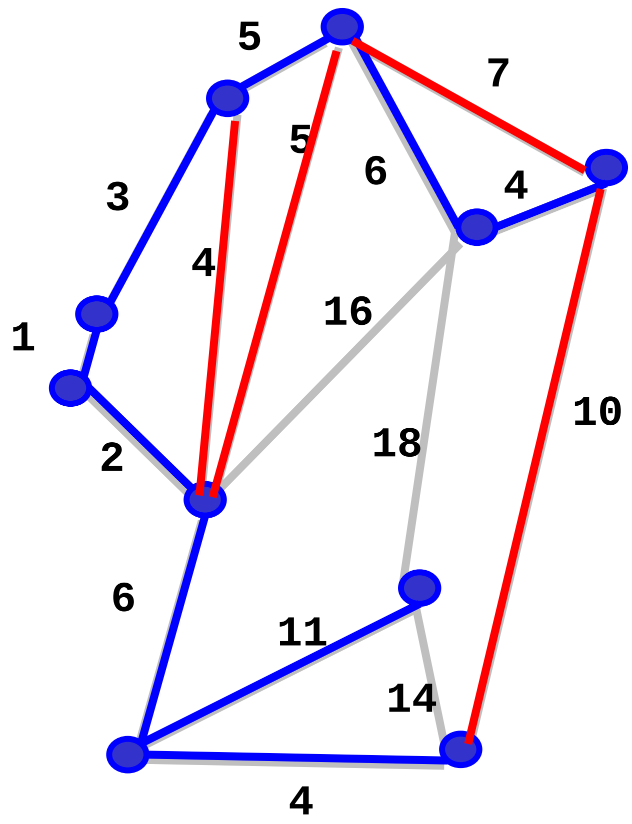


Kruskal's Algorithm

- Kruskal's Algorithm

  Start with an empty forest F. Continue "adding" edges e to F in order of increasing cost until F becomes a tree. Adding an edge e=(v, w) to F proceeds as follows:
  - *If F already contains a tree containing both v and w, then e is dropped*
  - *If no tree in F contains either v or w, then a new tree formed by e is added to F*
  - *If F contains a tree T containing either v or w and neither T nor any other tree in F contains the other node, then e is added to T*
  - *If F contains a tree T containing either v or w and a tree T' containing the other node, then T, T' and e are merged into one tree*

Example



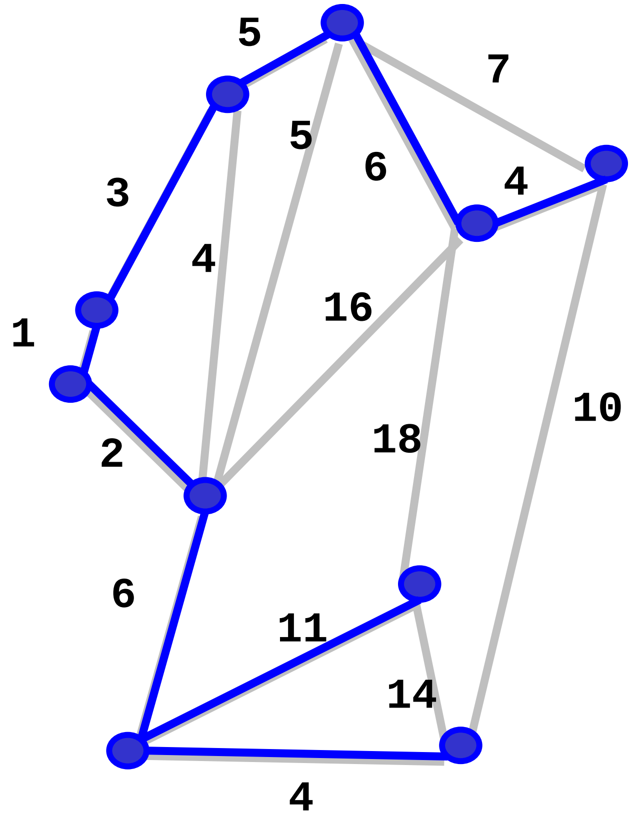
Proof

- By induction (only central idea)
  - We show that all trees in F are a MST of a subgraph of G
  - Claim is true at the beginning (F empty)
  - Assume claim holds when we consider the next edge e=(v, w)
  - Case 1: Claim holds, because e would introduce a cycle, and e has the highest cost on this cycle (all cheaper edges were considered before). Thus, e cannot be in an MST for G
  - Case 2: Claim holds because e is the cheapest edge connecting v and w, and thus the new tree is a MST (for v and w)
  - Case 3: Claim holds because e is the cheapest edge connecting v (or w) and T, and thus the new tree is a MST
  - Case 4: Claim holds because e is the cheapest edge connecting T and T', and thus the new tree is a MST

Boruvka‘s Algorithm

- Boruvka‘s Algorithm
  Start with an empty forest F. Add all edges (at once) that connect a node with its “cheapest” neighbor (edge with least cost) – taking care of not introducing cycles. Then consider each pair of trees in F in order of the cost of connection and add cheapest crossing-edge until F becomes a unique tree.
- Proof (and details) omitted; see [Sed04]

Example



Communalities

- All three algorithms iteratively choose an edge by the cut property or reject an edge by the cycle property
  - Prim: Growing T is one partition, all other nodes the other (isolated nodes)
  - Kruskal: Each T that grows is one partition, all other nodes the other (islands of mini-MSTs)
  - Boruvka: Each T that grows is one partition, all other nodes the other (islands of mini-MSTs)
- Differences
  - The order in which edges are chosen – there are always many candidates
  - The data structures that these algorithms need to maintain

Content of this Lecture

- Minimal Spanning Trees
- Basic Properties
- Algorithms
- Implementation
  - Prim's, Kruskal's

Implementing Prim‘s Algorithm

- ChooseCheapest: Choose cheapest edge from R connecting a node in T to a node not yet in T
- Brute force: Search all such edges in every step
- Better
  - Maintain a PQ of nodes reachable by one edge from T sorted by cost
  - When adding a new node to T, look at its neighbors and add them to the PQ (if not reachable before) or update costs (if now there is a cheaper edge reaching them)

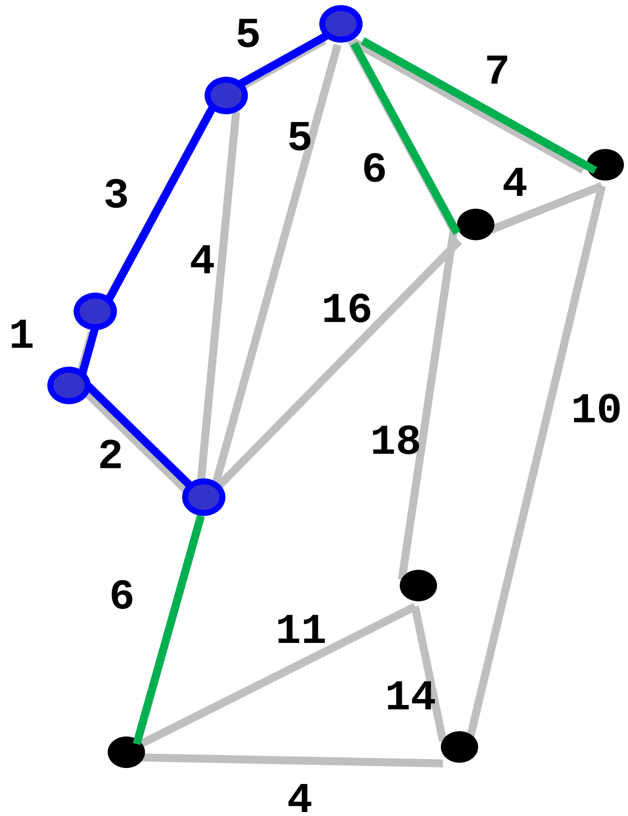
```
G := (V, E);
T := ∅;     # Growing T
R := E;     # Remaining edges
for i = 1 to |V|-1 do
  e := chooseCheapest( T, R);
  T := T ∪ e;
  R := R \ e;
end for;
```

Example

A
B
C
D
E
F

G
H

I,J



- T = {A, F, E, B, G}
- PQ = {(D,6), (I, 6), (C, 7)}

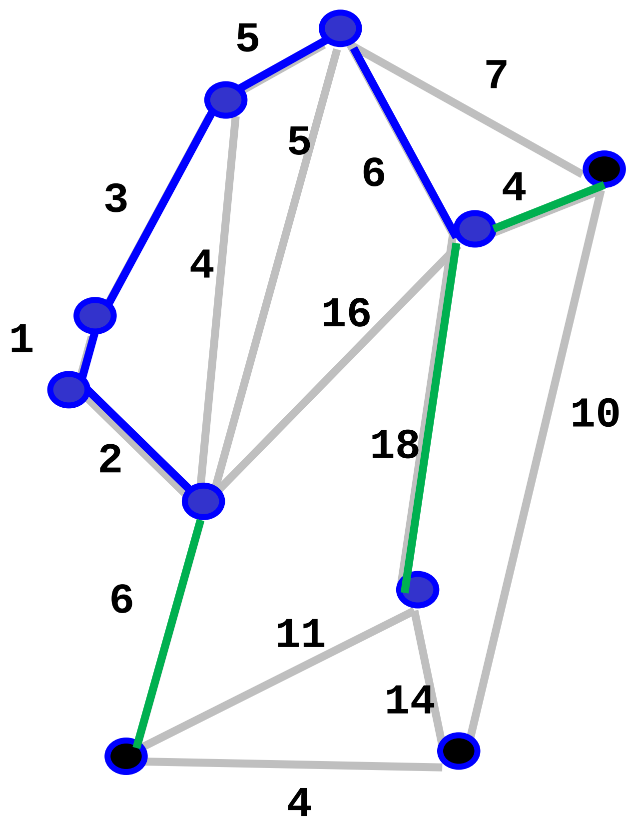
- Choose (A-D, 6)

Example

A
B
C
D
E
F

G
H

I,J



- T = {A, F, E, B, G}
- PQ = {(D,6), (I, 6), (C, 7)}

- Choose (A-D, 6)
- New T: {A, F, E, B, G, D}
- PQ = {(C,4), (I, 6), (H, 18)}

Complexity

- $n=|V|$, $m=|E|$
- Prim' algorithm runs in $O((n+m)*\log(n))$
  - n times through the loop, performing altogether at most m PQ-operations in $\log(n)$
- In dense graphs ($m \sim n^2$), this means $O(m*\log(n))$

Implementing Kruskal‘s Algorithm

- ChooseCheapest: Simply choose cheapest edge in E
  - I.e., sort E at the beginning
- This is called a UNION-FIND data structure
  - Maintains a set of sets (all trees T)
  - Needs a method for quickly finding the set containing a given element (find)
  - Needs a method for quickly merging two sets (union)
- Can be implemented in O(m*log(n))

```
G := (V, E);
F := ∅;
repeat
  (v,w) := chooseCheapest( E);
  E := E \ (v,w);
  T := find( v);
  T’ := find (w);
  if T=T’=∅ then
    F.add( {(v,w)});
  else if T’=∅ then
    T.add ( {v,w});
  else if T=∅ then
    T’.add ( {v,w});
  else if T≠T’ then
    T := T ∪ T’;
  end if;
until |T|=|V|;
```

Exemplary Examination Questions

- Correctly formulate and prove the Cut-property, a tool for computing MSTs
- Compute a MST for the following graph ... using Prim's algorithm. After each step, show the sets T, R, and the sate of the priority queue Q
- Prove or falsify: If all edge weights of a graph G are pairwise distinct, then G has only one MST
- Prove or falsify the correctness of the following algorithm for computing an MST for a graph G:
  - (1) Set G'=G;
  - (2) If G' contains no cycle, return G' as MST;
  - (3) Otherwise, chose an arbitrary cycle in G' and remove the edge with the highest weight on this cycle; then goto 2