Ticket to Ride: Concepts in Graph Theory

Patrick Schäfer
patrick.schaefer@hu-berlin.de

Semesterprojekt: Implementierung eines Brettspiels, WS 18/19

Agenda

- Last week
  - Technical refinement for new user stories
  -
- Today (w/ POs), 13:30
  - Sprint #1 Review Meeting; bring a laptop & presentable prototype
  - Sprint #2 kickoff; present your Sprint Backlog
  - short talk on
    - Ticket to Ride: concepts in graph theory
    - Coding guidelines / conventions

The Board Represents a Graph

Ticket to Ride and Graph Theory

- G = (V,E), V=Cities, E = Railways.

- Each vertex of the graph represents one city in Europe
- An edge connects two cities
- Each edge has a color and a length (cost)
- The graph has more edges than any player can claim
- The set of cities and edges is a player's *edge-induced* subgraph
- *Connected components: Player's s*ubgraphs don't have to be connected
- Paths and Cycles
  - A destination ticket is met, when there is a path between the two cities
  - Creating cycles do not increase coverage, and thus do not help meeting destination tickets (but may block other players)

Graph Algorithms

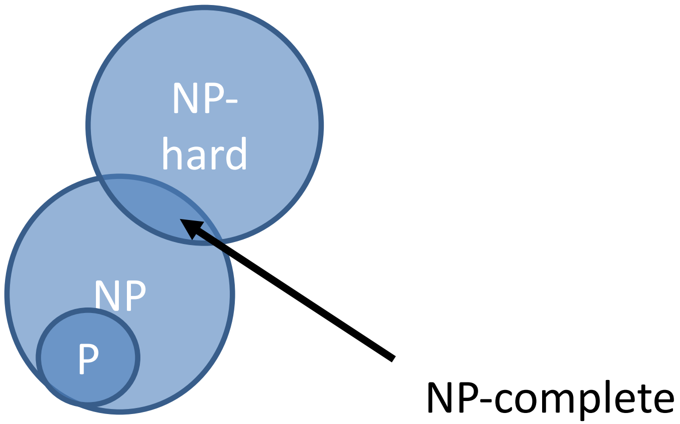
- Concepts known from: “Algorithmen und Datenstrukturen”
  - Graph representations: Adjacency List, Adjacency Matrix
  - Shortest paths: Dijkstra, Floyd Warshall
  - Graph traversal: BFS, DFS
  - Minimum Spanning Tree: Prim, Kruskal
  - Topological sorting of directed graphs.

Ticket to Ride Rules and their Concepts in Computer Science

- What is the best representation of the board?
  - *Adjacency matrix or adjacency list?*
- Given a destination ticket, what is the shortest path?
  - *Shortest path: Dijkstra*
- How to fulfill destination tickets with the least amount of trains?
  - *Minimum spanning tree on subgraph (Minimum Steiner tree)*
- Calculating the final score:
  - List of routes claimed by a player
    - *Lookup in graph data structure (adjacency matrix or adjacency list)*
  - List of destination tickets fulfilled by a player.
    - *Graph traversal: DFS / BFS*
  - 10 point bonus is awarded to player with the longest route on the board.
    - *Longest path in a tree / graph*

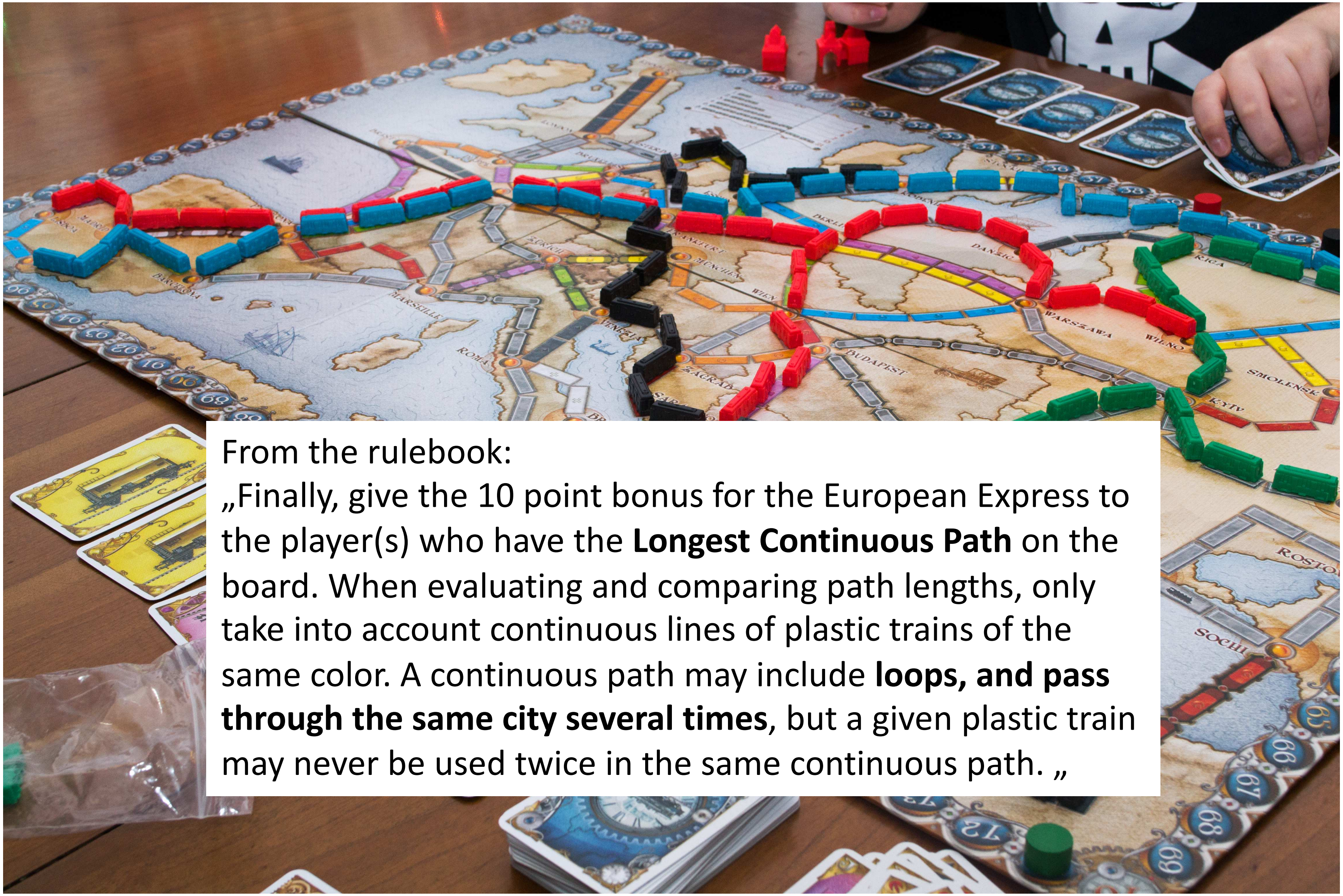
P, NP, NP-hard, NP-complete

- Definition:
  - P is the set of **decision problems** that can be solved in polynomial time
  - NP is the set of **decision problems** where we can **verify** a solution in polynomial time
  - NP-hard: at least as hard as NP (using polynomial time reduction)
  - NP-complete: it is NP-hard and in NP



What the world might look like…

Longest Simple Path

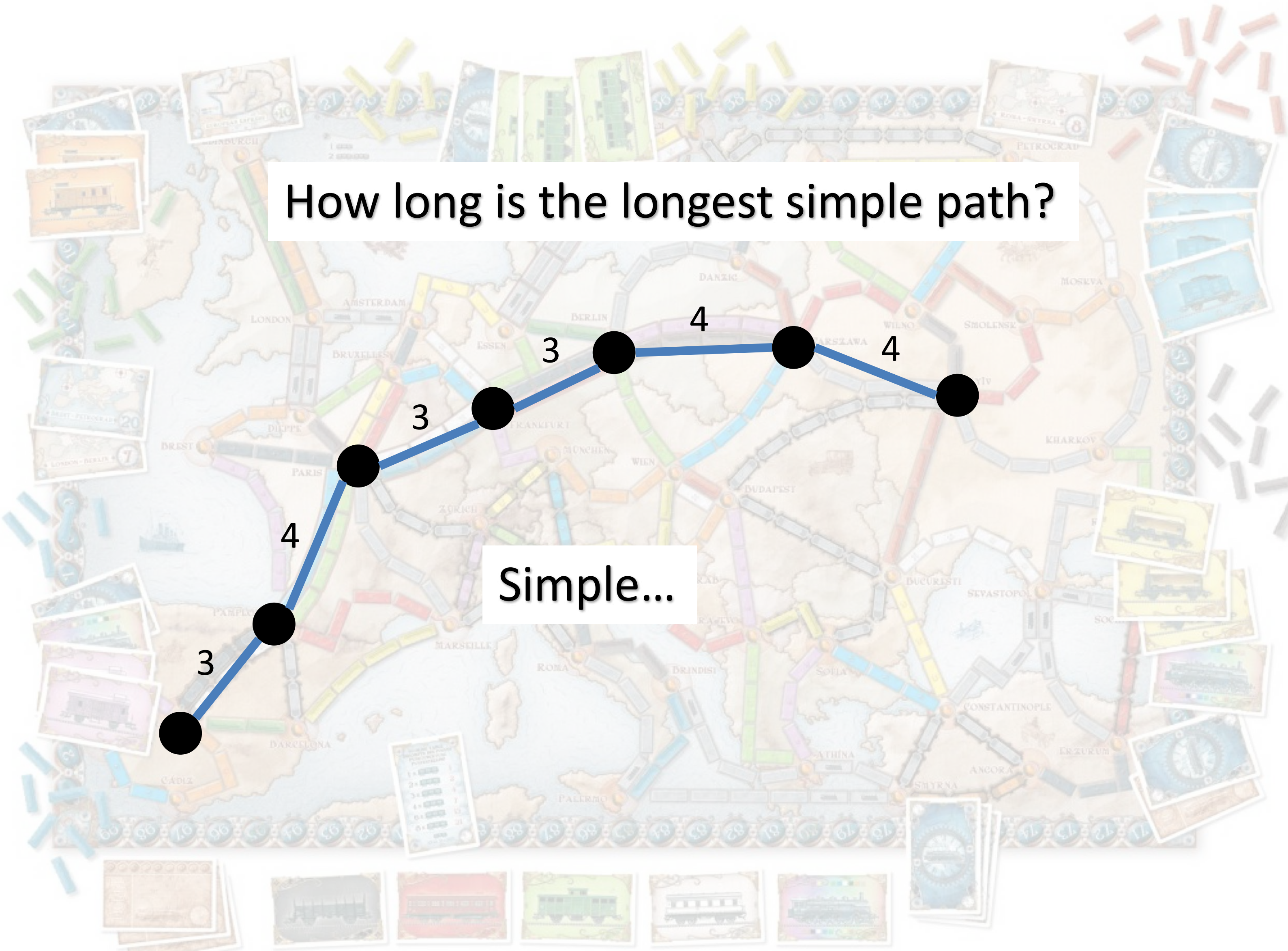


From the rulebook:

„Finally, give the 10 point bonus for the European Express to the player(s) who have the **Longest Continuous Path** on the board. When evaluating and comparing path lengths, only take into account continuous lines of plastic trains of the same color. A continuous path may include **loops, and pass through the same city several times**, but a given plastic train may never be used twice in the same continuous path. „

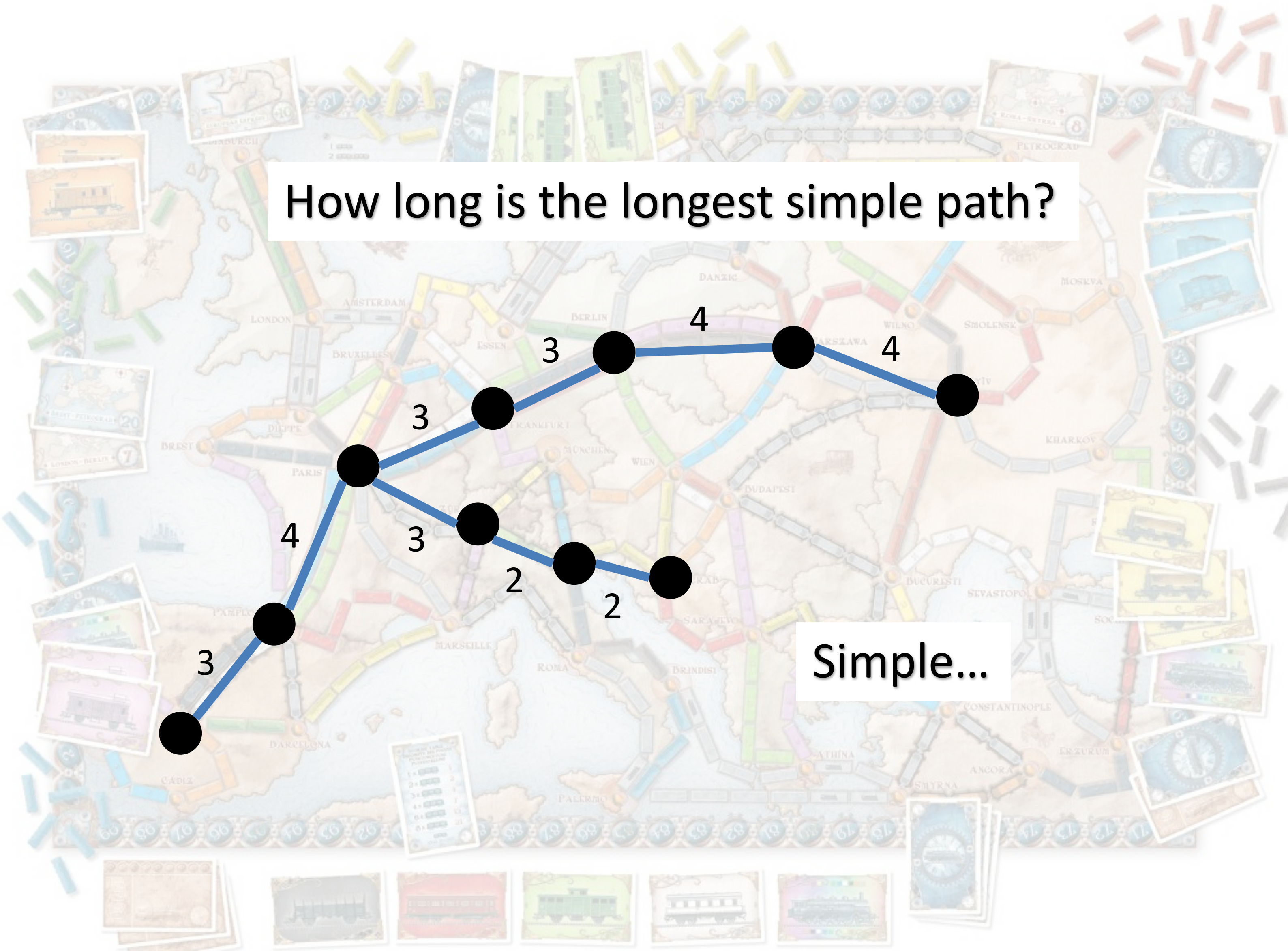
Acyclic Graph / Undirected Tree

Acyclic Graph / Undirected Tree



Acyclic Graph / Undirected Tree

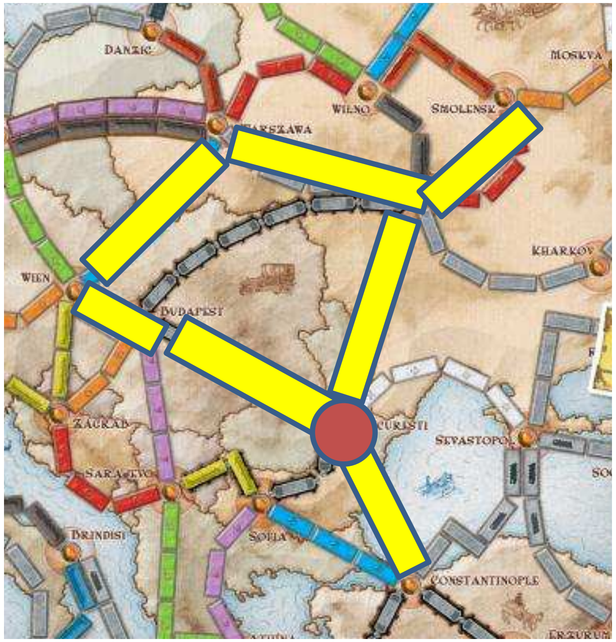
Acyclic Graph / Undirected Tree



Longest Simple Path

- There is an algorithm for finding the longest simple path in undirected trees using two Depth-First-Searches:
  - Start DFS from a random vertex *v* and find the farthest vertex *v′* away.
  - Now, start a DFS from *v′* to find the vertex *v′′* farthest away from it. This path is the longest path in the graph.

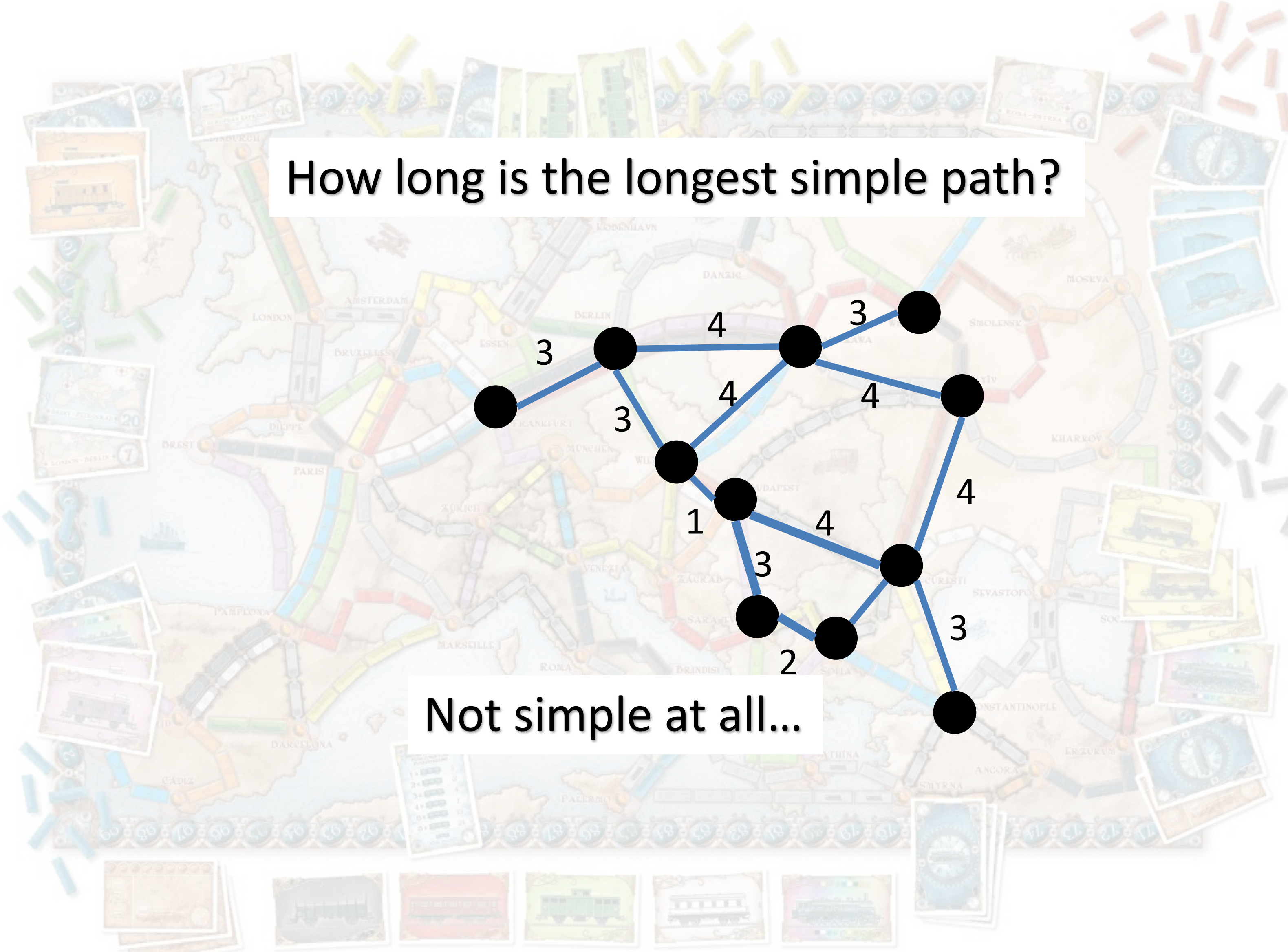
- Does this still work in a cyclic graph?



Cyclic, Undirected Graph

Cyclic, Undirected Graph

How long is the longest simple path?



Not simple at all…

Reformulation

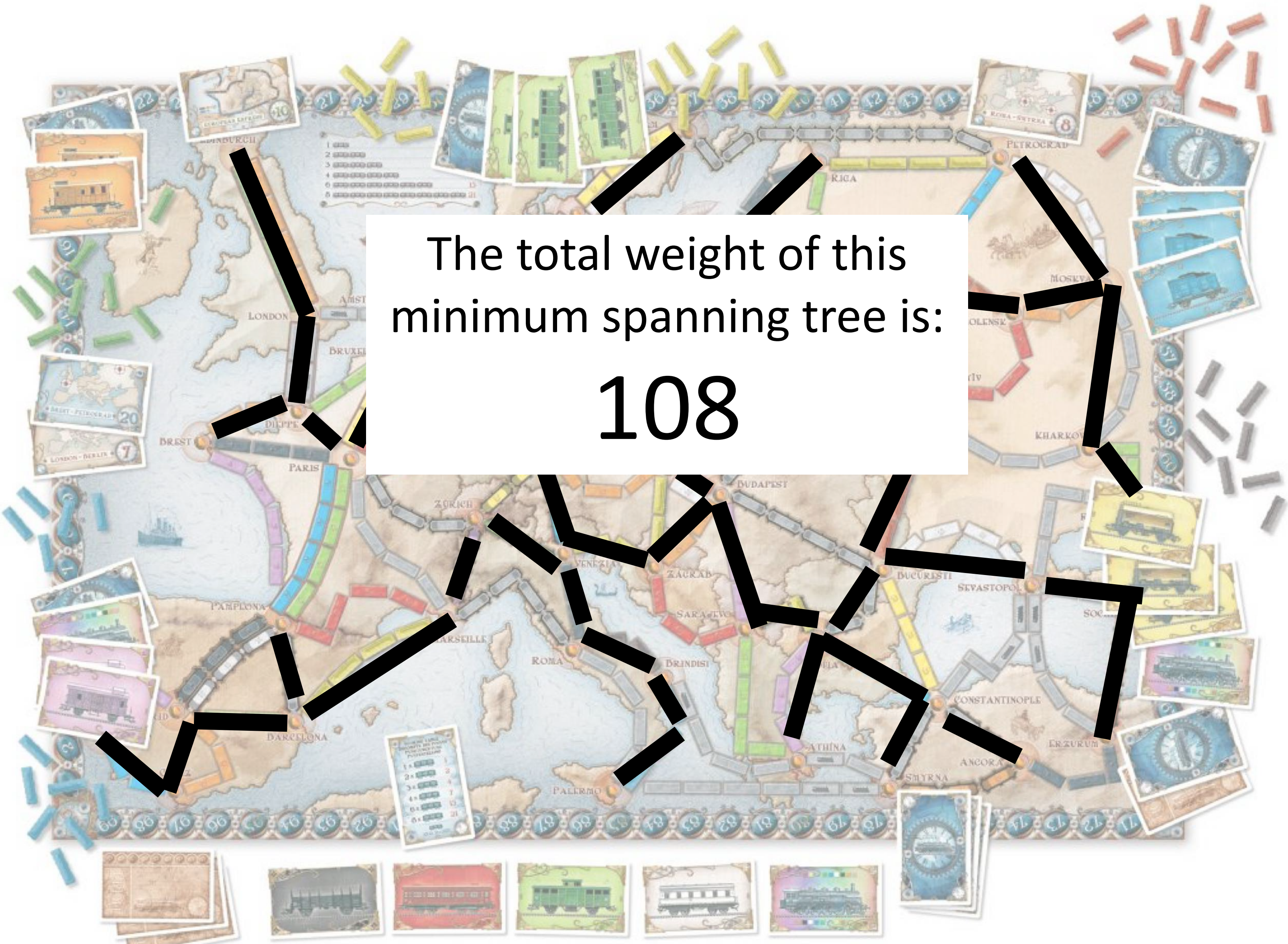
- Is there a path in the player's edge-induced subgraph, that visits all edges? NP-hard
  - Check for Euler path of the full graph is in P
  - But if there is no Eulerian path, we have to check O(2^n) many subsets

- Finding the longest simple path in a cyclic graph is NP-hard. Thus, there is likely to be no polynomial time algorithm.

- There are approximate algorithms in polynomial time.
- For final scoring, we need the **exact length** of the longest path (not an approximation).

- Side note: finding the longest simple path in an undirected tree (acyclic graph) is in polynomial time.

A MST for Ticket to Ride



The total weight of this minimum spanning tree is:

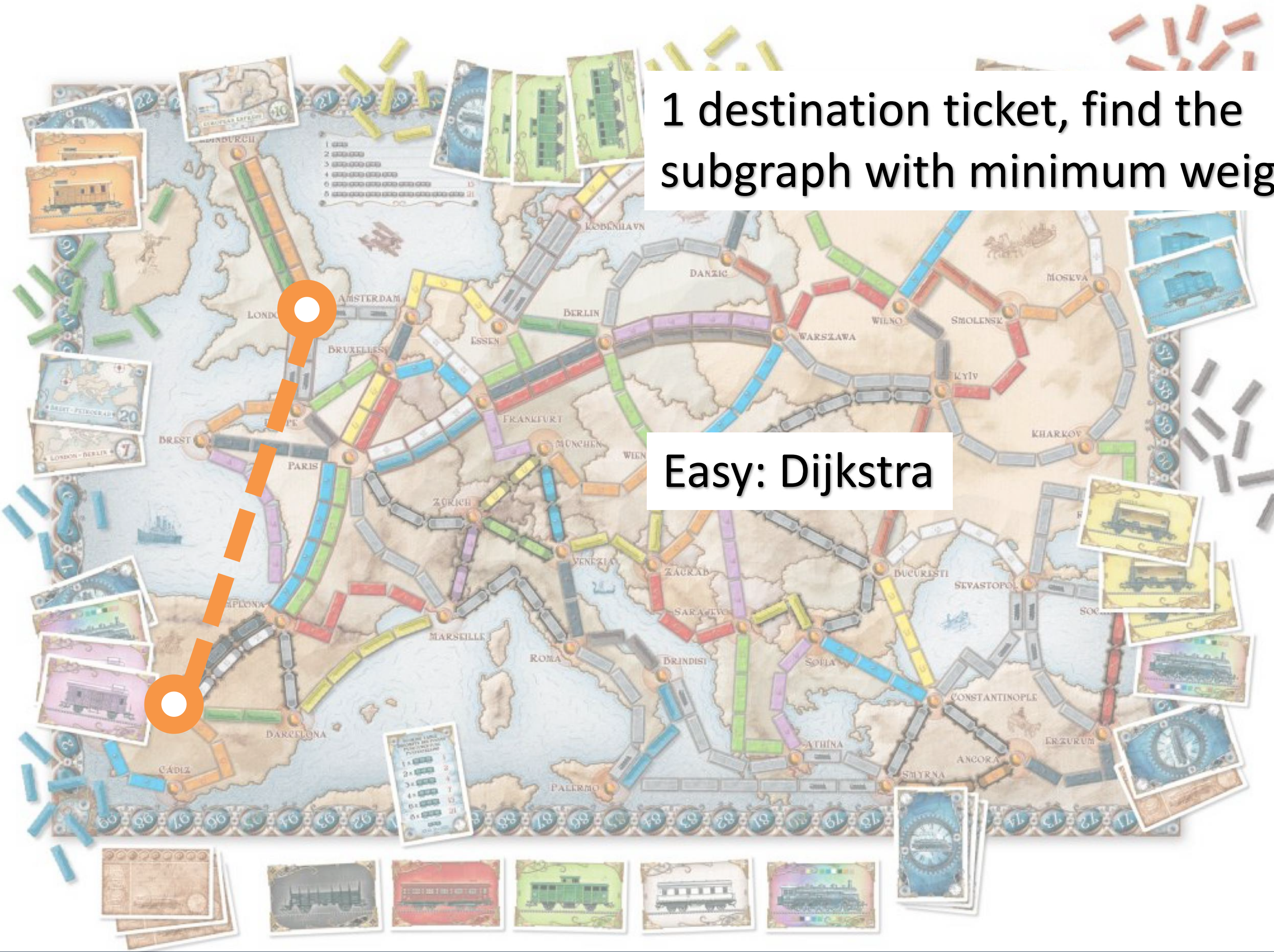
108

Minimum Spanning Tree (Forest)

- A spanning tree of the full graph would guarantee that any destination ticket is fulfilled.
- But payers do not have enough train tokens to claim a spanning tree of the full graph (45 vs 108).
- Thus, the best strategy is to capture a minimum spanning tree or forest of a subset of vertices (based on the destination tickets).

- **Steiner Tree / Forest:** Given an undirected, weighted graph G=(V,E) and a subset of vertices V’, referred to as terminals, we search the subgraph G’ with minimum weight, that connects all terminals (and may include additional vertices).

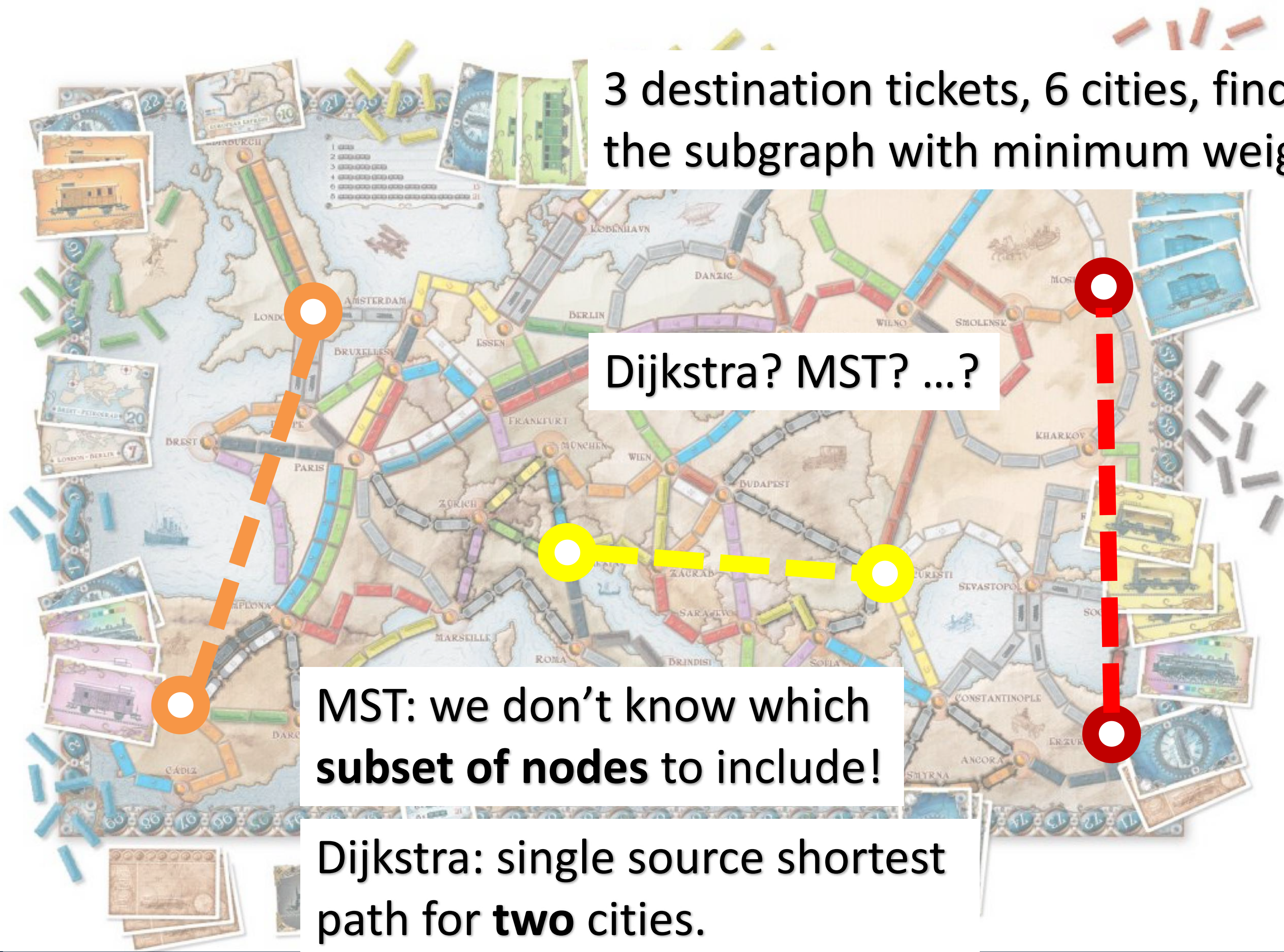
Shortest Path on Destination Ticket



1 destination ticket, find the subgraph with minimum weight.

Easy: Dijkstra

Minimum-Weight Subtree on Destination Tickets



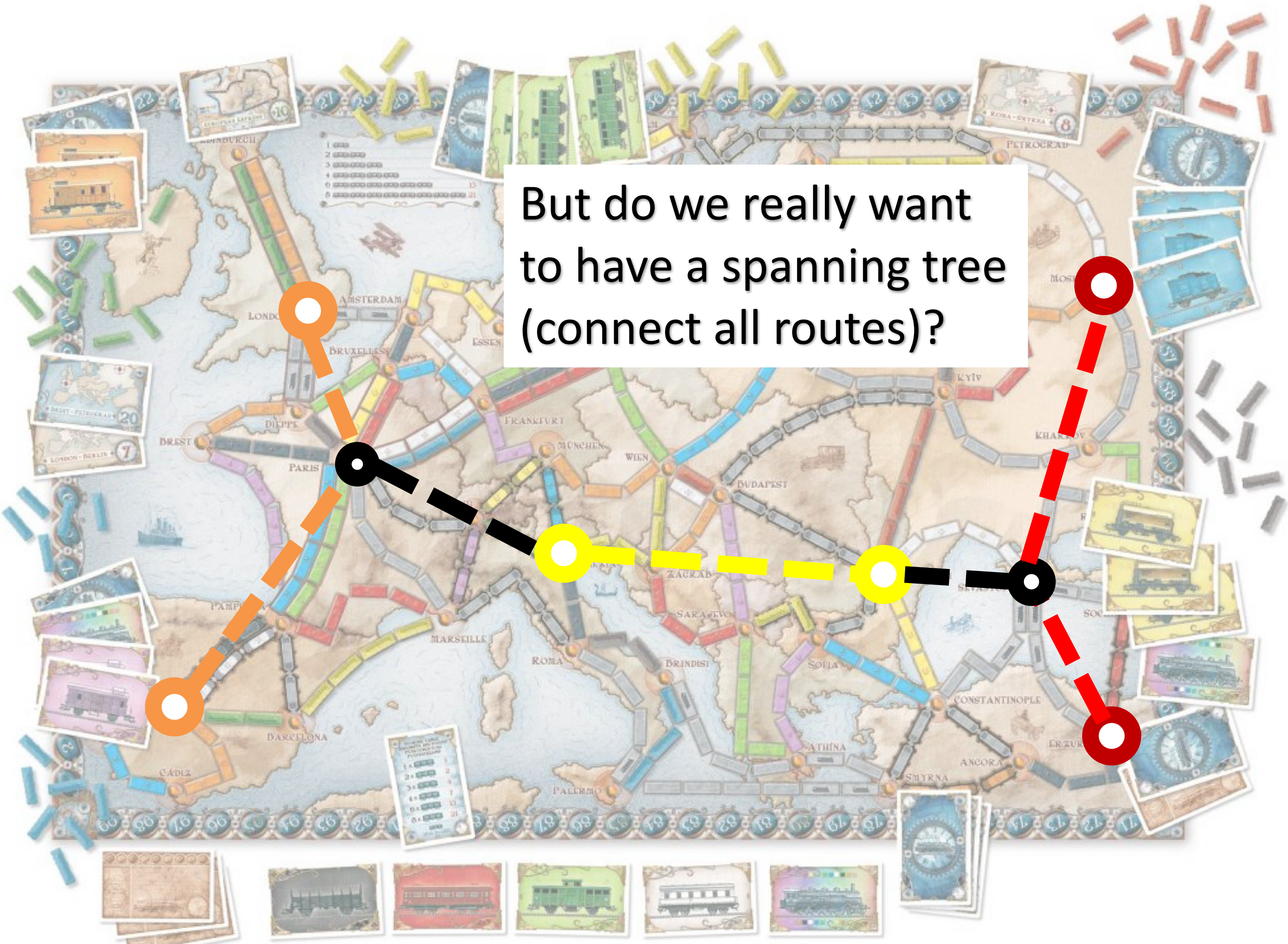
3 destination tickets, 6 cities, find the subgraph with minimum weight.

Dijkstra? MST? …?

MST: we don’t know which **subset of nodes** to include!

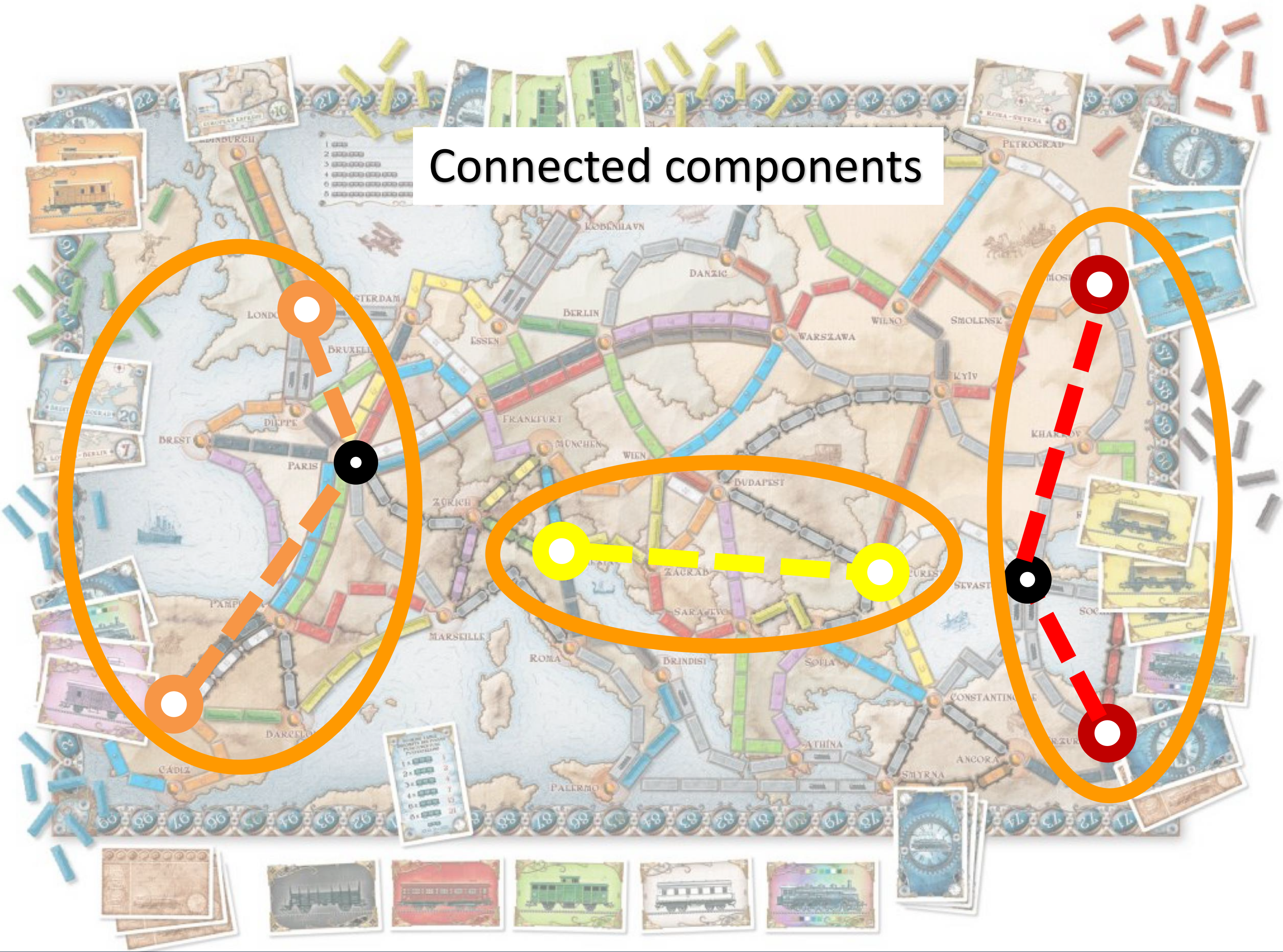
Dijkstra: single source shortest path for **two** cities.

Minimum-Weight Subtree on Destination Tickets

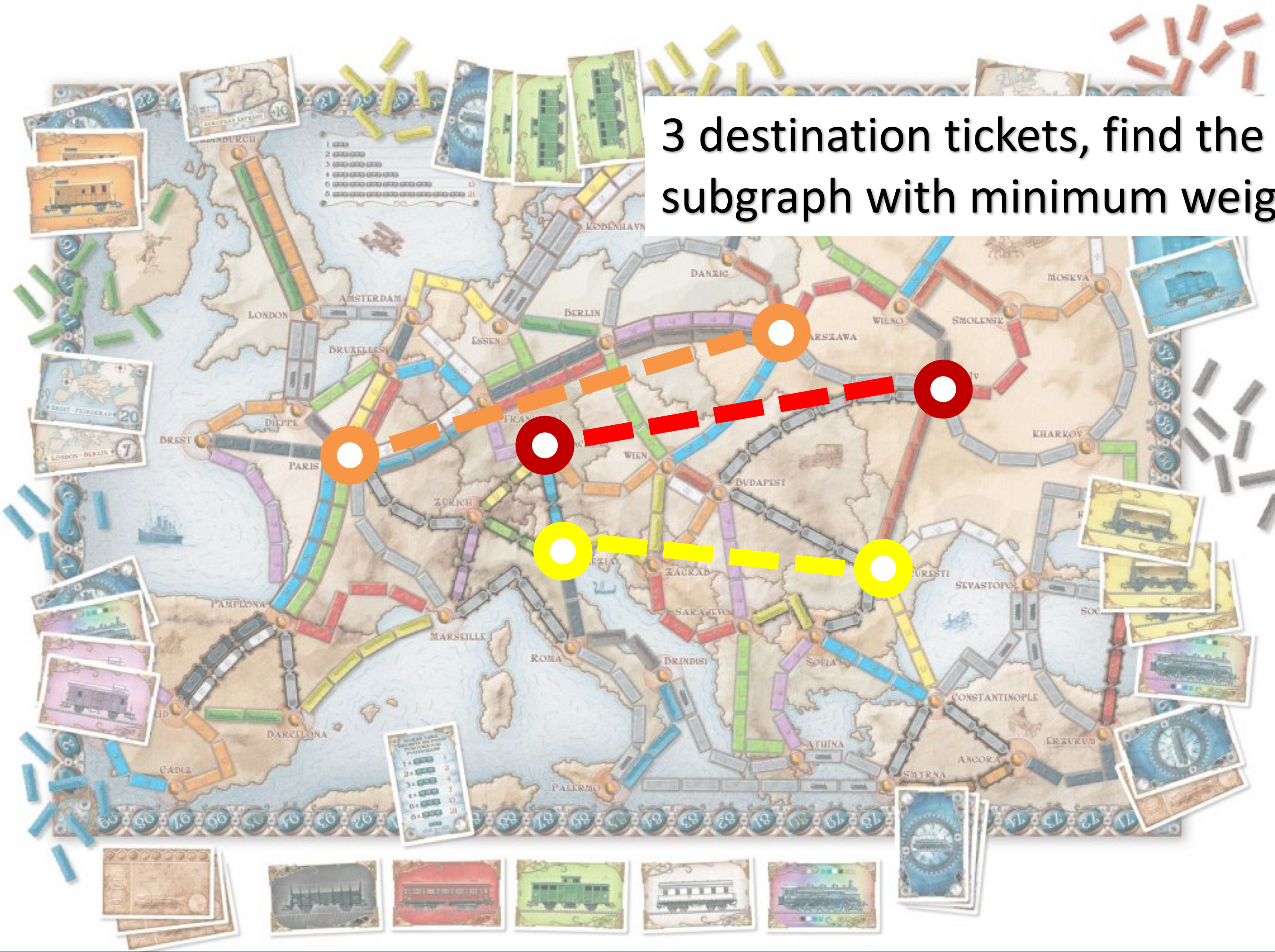


But do we really want to have a spanning tree (connect all routes)?

Minimum-Weight Subtree on Destination Tickets



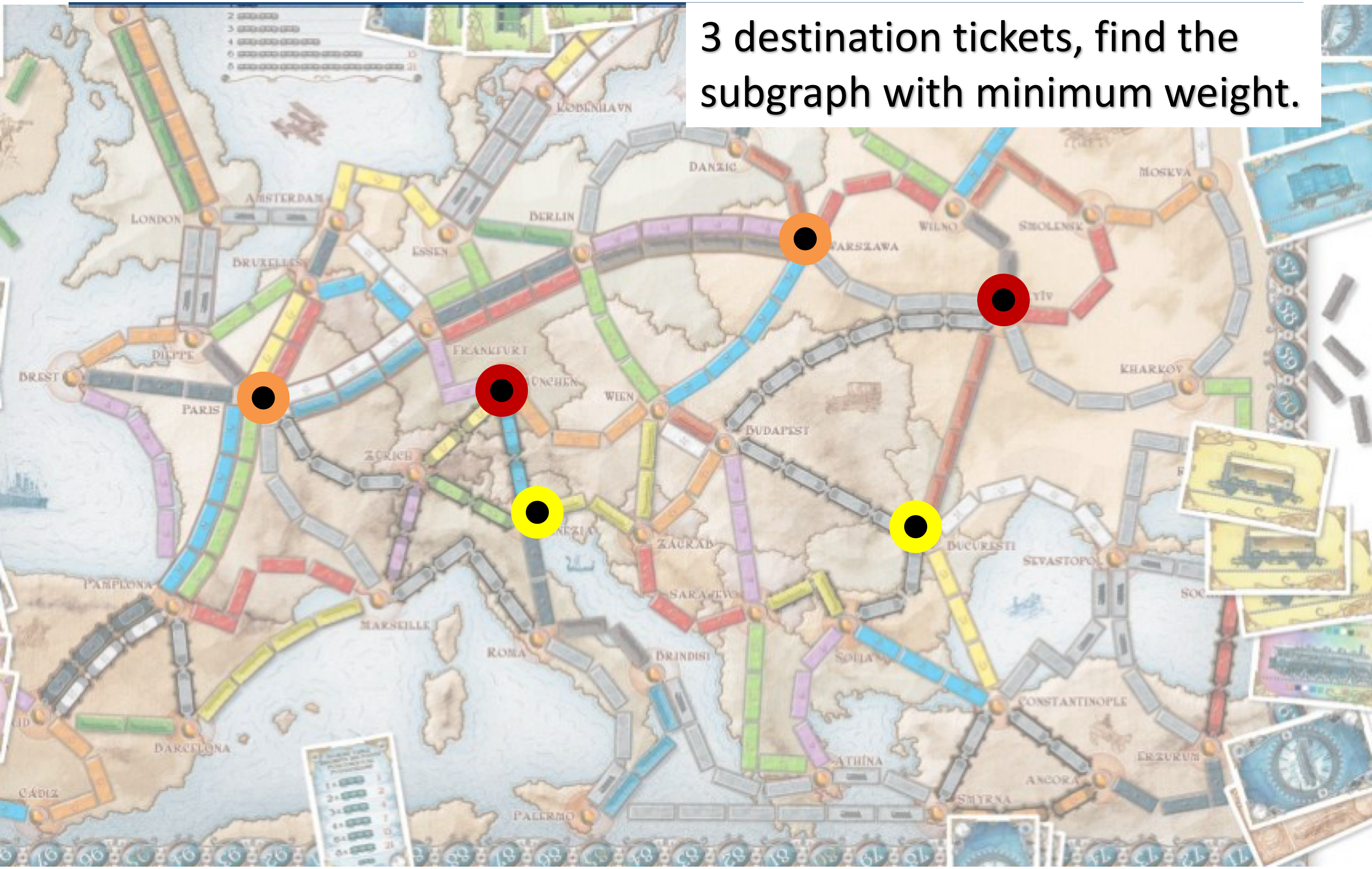
Minimum-Weight Subtree on Destination Tickets



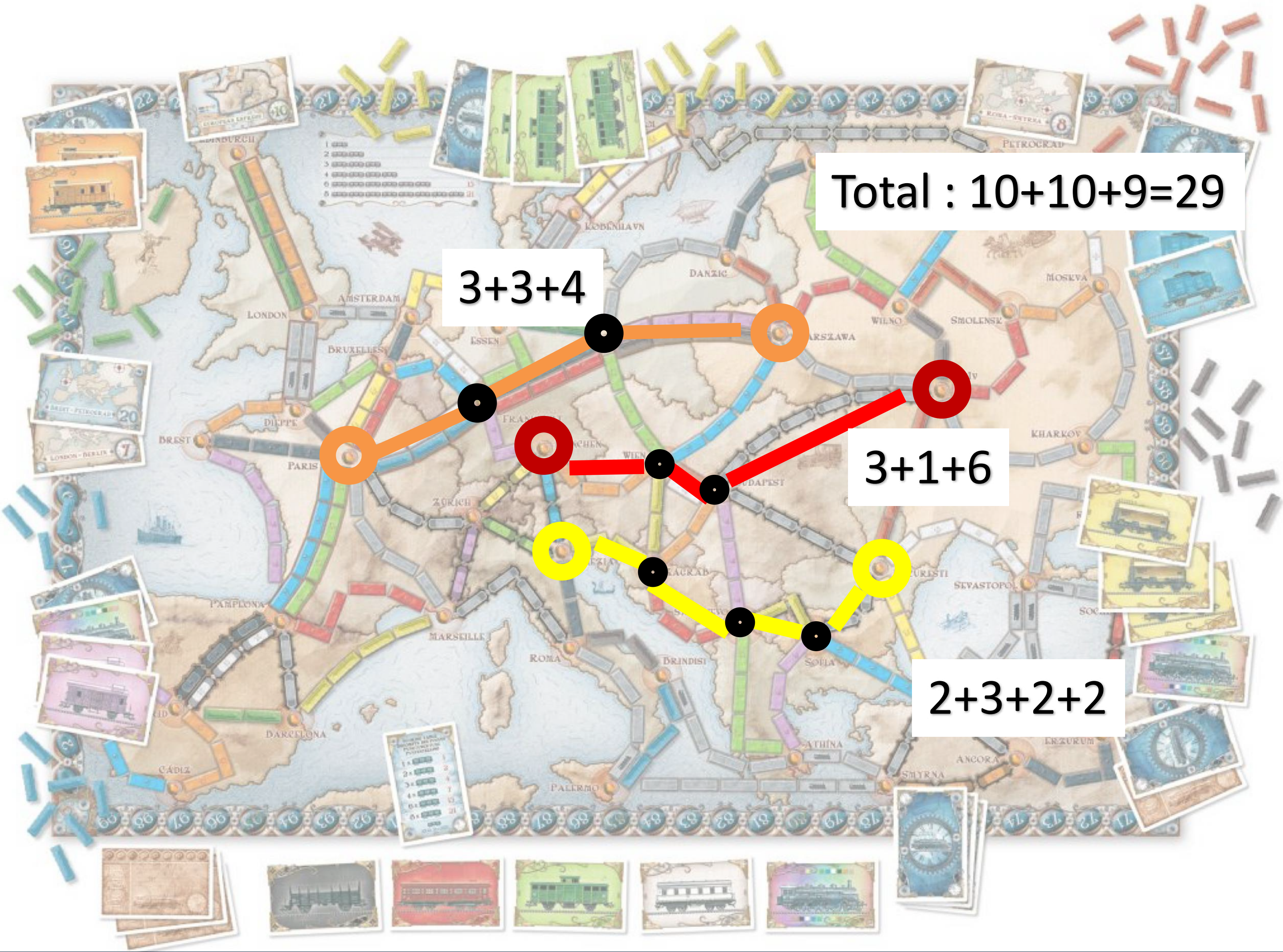
3 destination tickets, find the subgraph with minimum weight.

Minimum-Weight Subtree on Destination Tickets

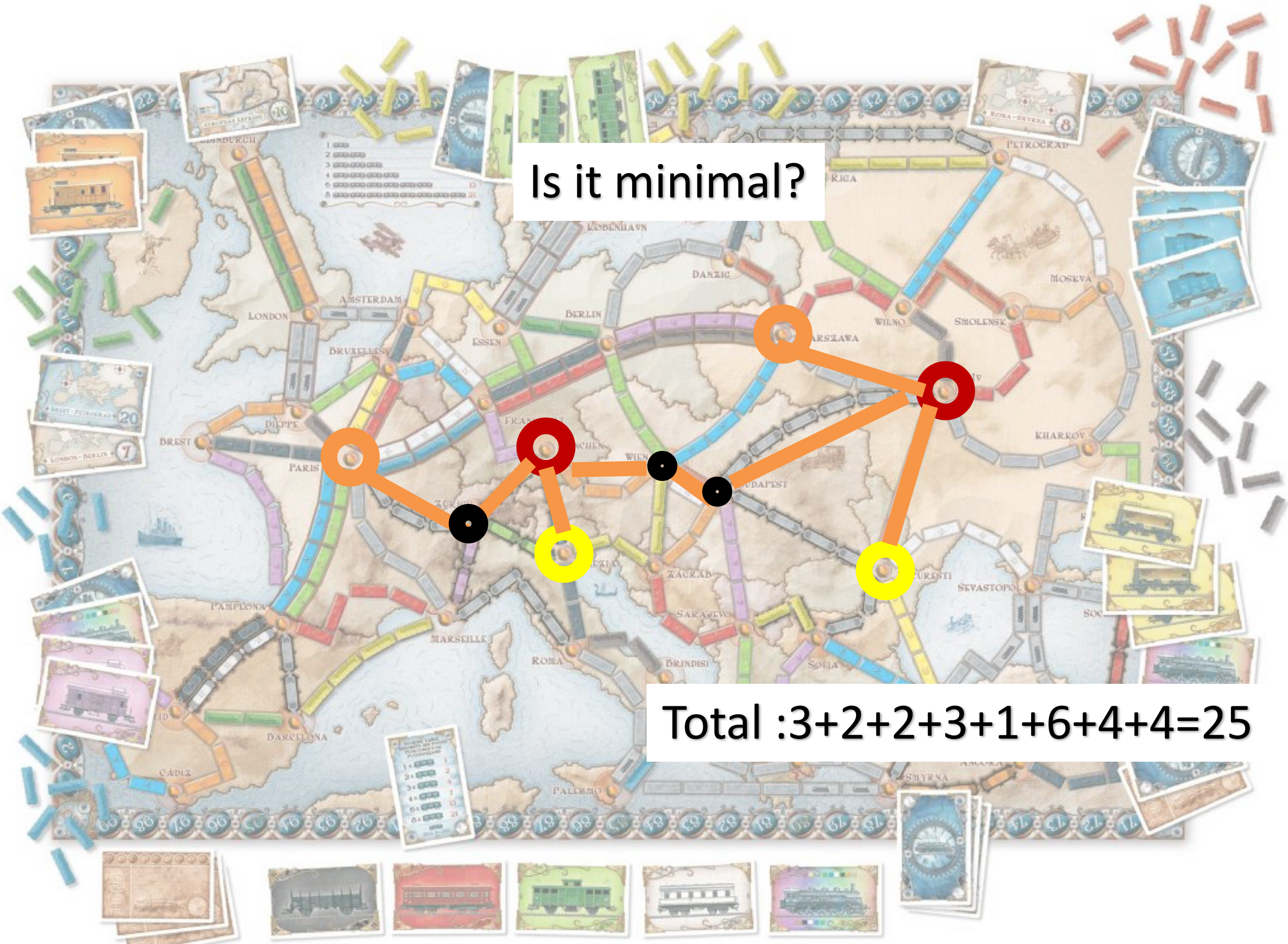
3 destination tickets, find the subgraph with minimum weight.



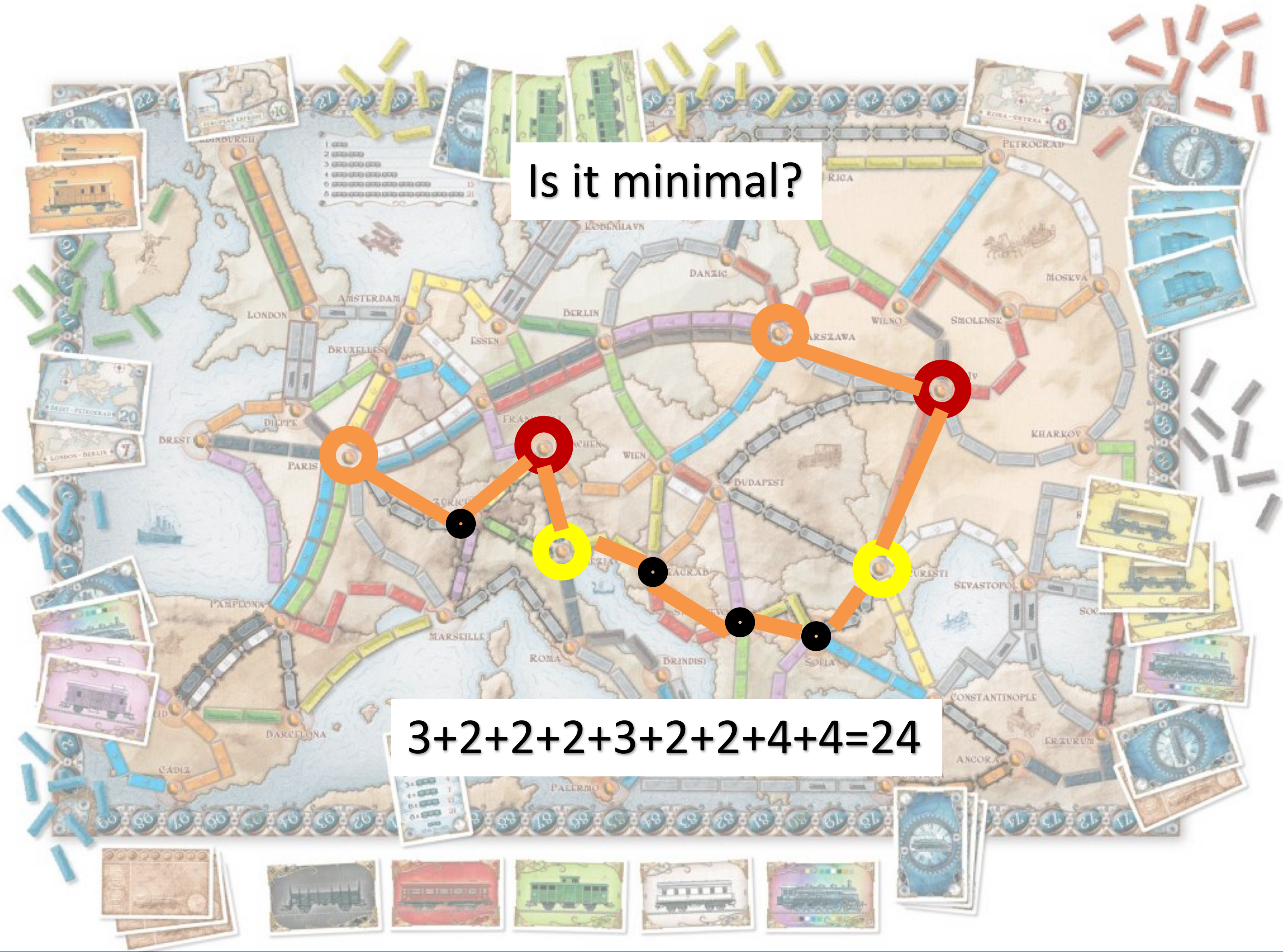
Using Dijkstra…?



A Spanning Tree on the Subgraph…



A Spanning Tree on the Subgraph…



A Spanning Tree on the Subgraph…

Is it minimal?

3+3+4+4+4+2+2=22

NP-hardness

- We are dealing with NP-hard **optimization problems** [1]:
  - „**LONGEST PATH**: Given a non-negatively weighted graph *G* and two vertices *u* and *v*, what is the longest simple path from *u* to *v* in the graph? A path is *simple* if it visits each vertex at most once.„
  - „**STEINER TREE**: Given a weighted, undirected graph *G* with some of the vertices marked, what is the minimum-weight subtree of *G* that contains every marked vertex? If *every* vertex is marked, the minimum Steiner tree is just the minimum spanning tree; if exactly two vertices are marked, the minimum Steiner tree is just the shortest path between them.„

[1] Garey and Johnsons, „Computers and Intractability: A Guide to the Theory of NP-Completeness“

Steiner Tree / Forest in Cyclic Graphs is NP-hard

- Steiner Tree optimization problem is NP-hard, thus there is likely to be no exact polynomial time algorithm.

- Naïve approach:

```
for each subset of nodes:                      // 2^|V|-Subsets
        compute the MST.                       // O(|E|+|V|log|v|)
pick the subset with minimum costs.
```

- There are *heuristic* algorithms with polynomial time, that have upper bound guarantees on the maximum cost.
- Finding a good algorithm is part of the AI-Challenge.

Literature

- 21 NP-Hard Problems: http://web.engr.illinois.edu/~jeffe/teaching/algorithms/2009/notes/21-nphard.pdf
- Taking Students Out for a Ride: Using a Board Game to Teach Graph Theory: http://www.cs.xu.edu/csci390/13s/p367-lim.pdf
- *Garey and Johnsons, „Computers and Intractability: A Guide to the Theory of NP-Completeness“*