# Algorithms and Data Structures

## (Search) Trees

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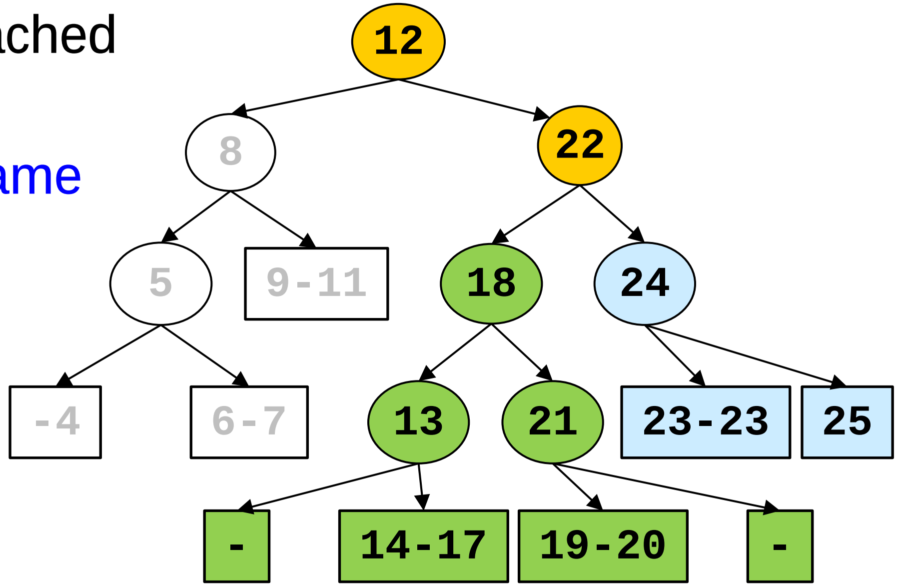
Source: whmsoft.net/

# Content of this Lecture

- Trees
- Search Trees
- Natural Trees

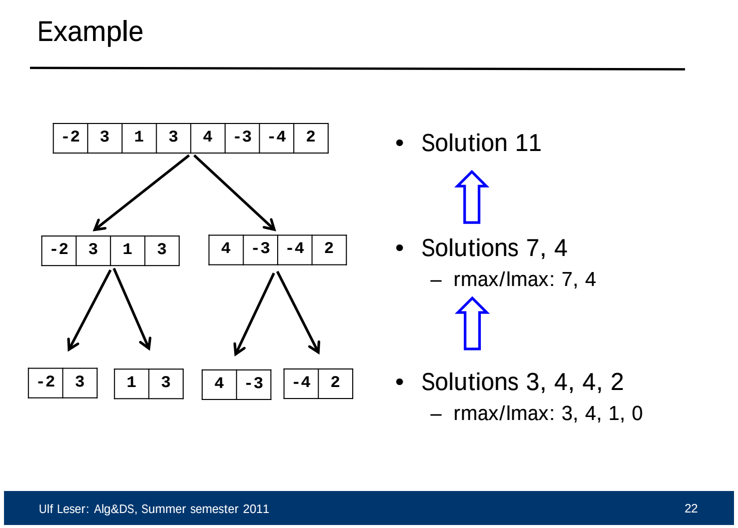
# Motivation

- In a list, (almost) every element has one predecessor / successor
- In a tree, (almost) every element has one predecessor but many successors
- These splits partition the set of all elements of the list
  - Every node in a tree can be reached by only one path from root
  - Partitions: All nodes with the same prefix in their access paths
  - Prominent criterion: Order
    - Elements with higher rank to left subtree, with lower rank to the right subtree

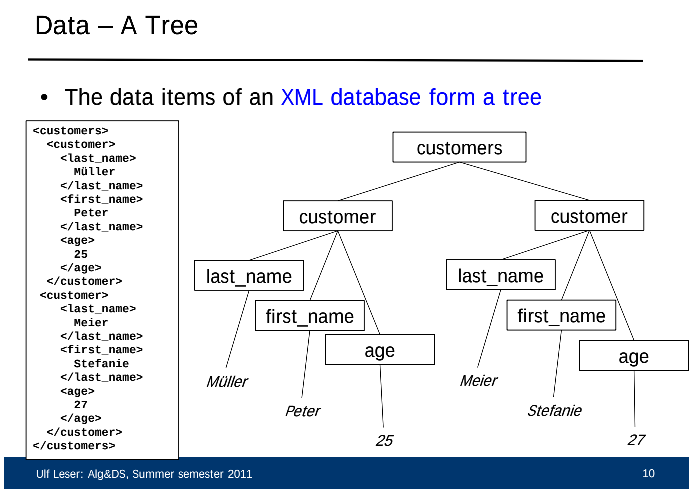


# Trees are everywhere in computer science

- Divide-and-conquer call stacks
  - Max-subarray
  - Merge-Sort
  - QuickSort
  - ...

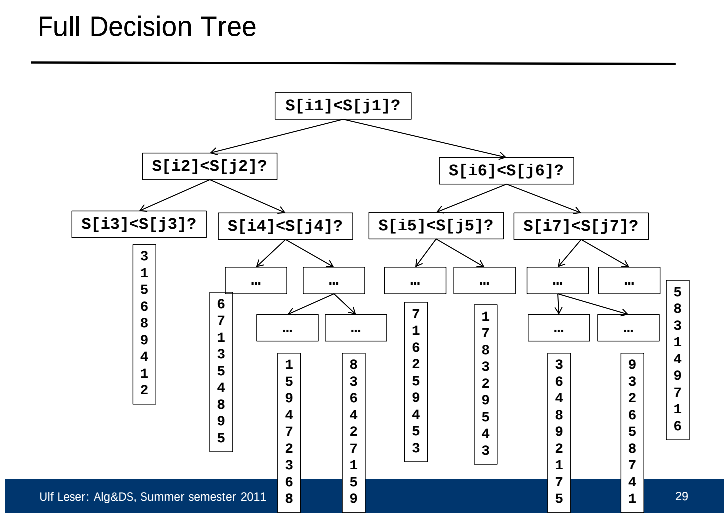



- XML
  - depth-first vs breadth-first traversal

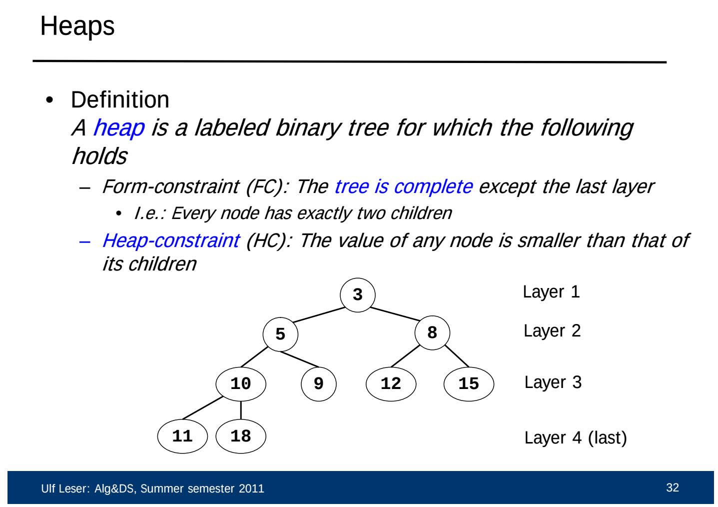


# Already Seen

- Decision trees for proving the lower bound for sorting




- Heaps for priority queues



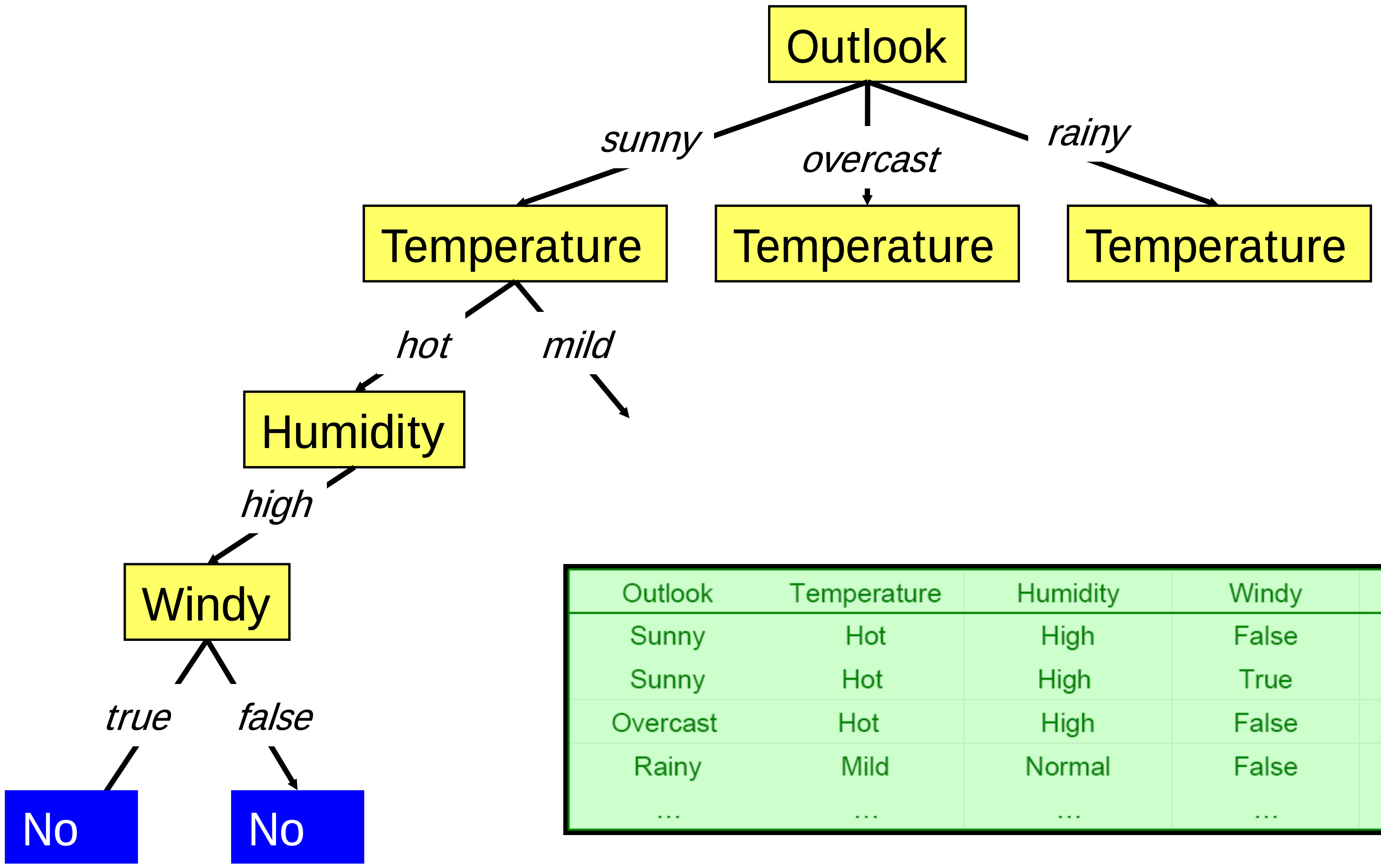

- …

# Machine Learning

- Want to go to a football game?
- Might be canceled – depends on the whether
- Let‘s learn from examples

| Outlook | Temperature | Humidity | Windy | Play |
|---|---|---|---|---|
| Sunny | Hot | High | False | No |
| Sunny | Hot | High | True | No |
| Overcast | Hot | High | False | Yes |
| Rainy | Mild | Normal | False | Yes |
| … | … | … | … | … |

# Decision Trees




| Outlook | Temperature | Humidity | Windy | Play |
|---|---|---|---|---|
| Sunny | Hot | High | False | No |
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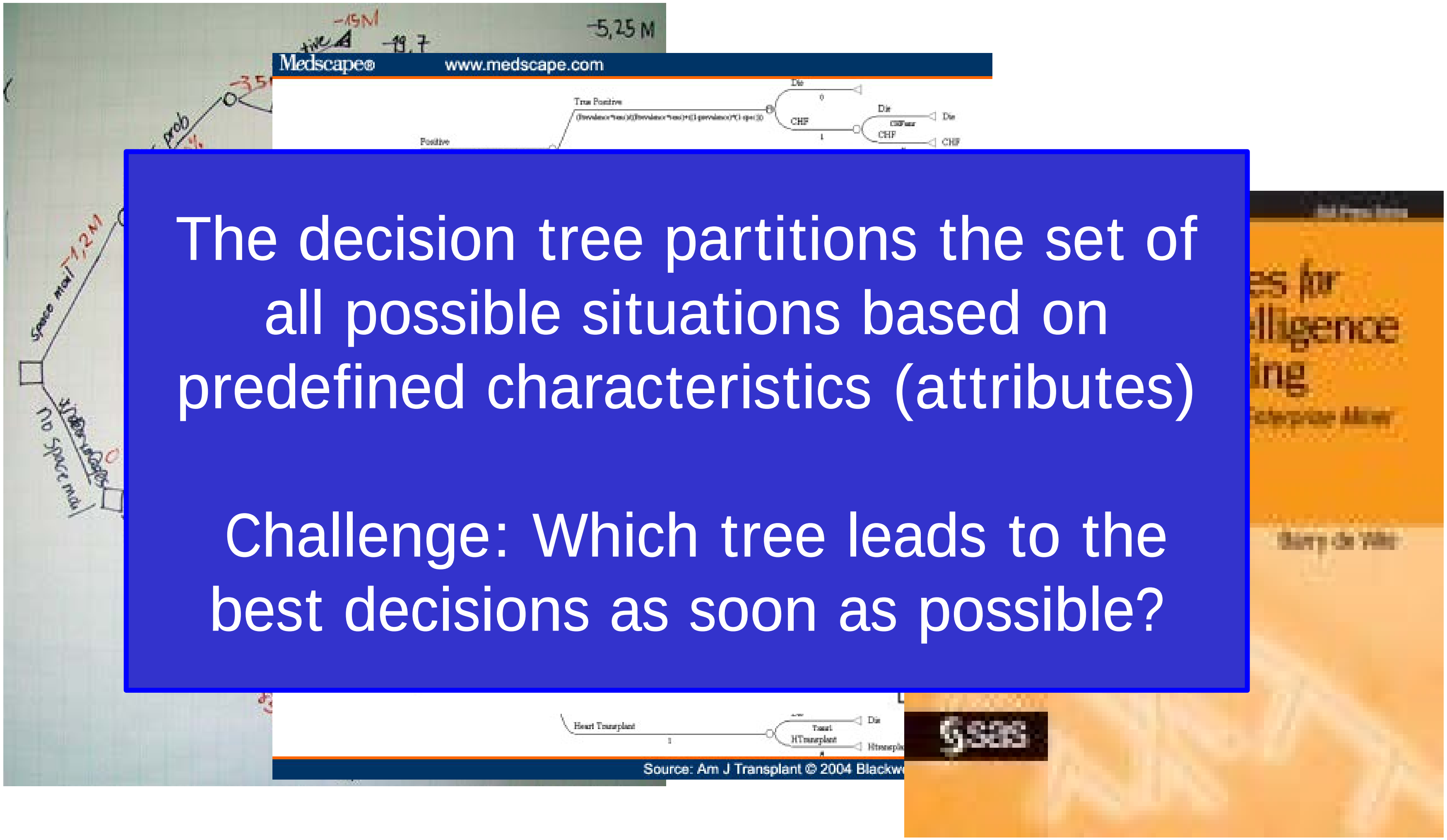
| Overcast | Hot | High | False | Yes |
| Rainy | Mild | Normal | False | Yes |
| … | … | … | … | … |

# Many Applications




The decision tree partitions the set of all possible situations based on predefined characteristics (attributes)

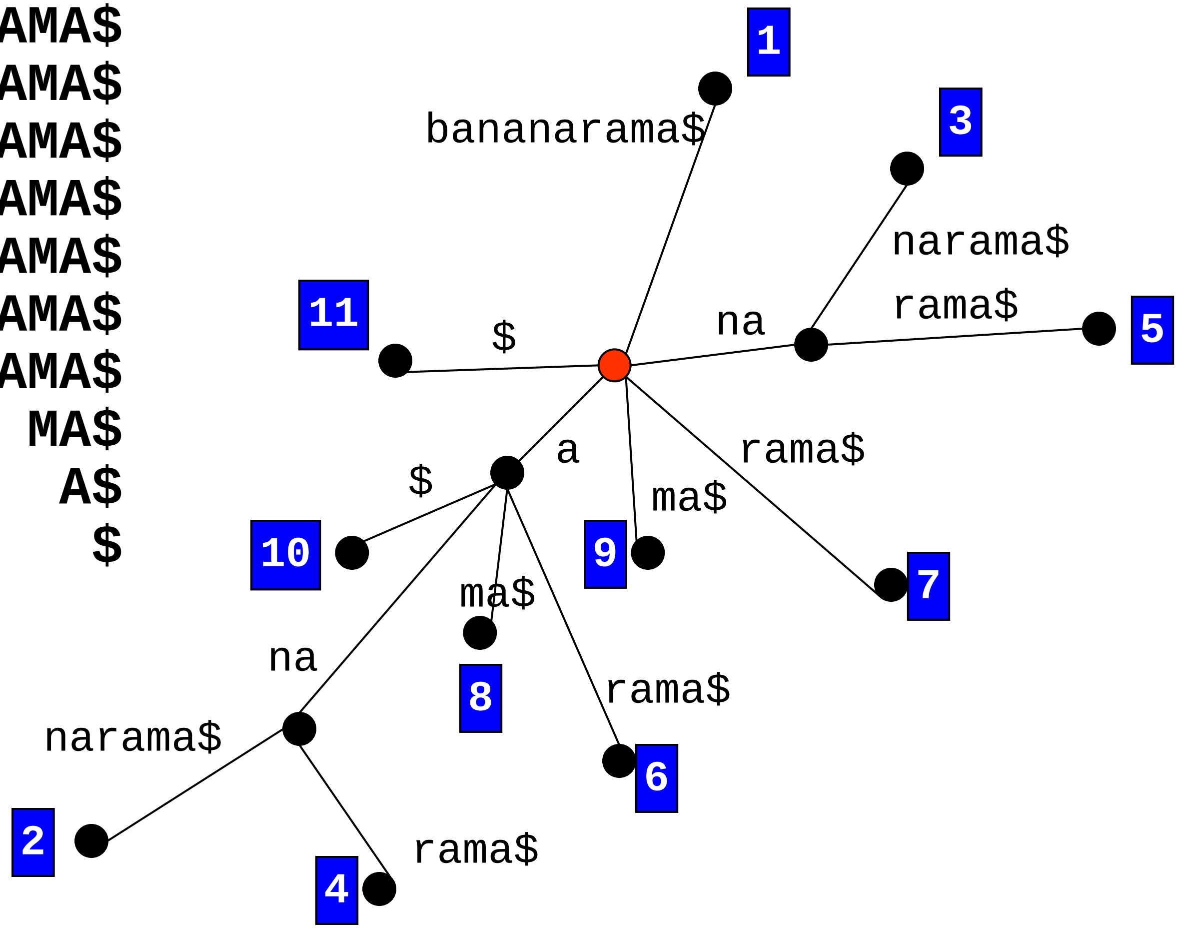
Challenge: Which tree leads to the best decisions as soon as possible?

# Suffix-Trees

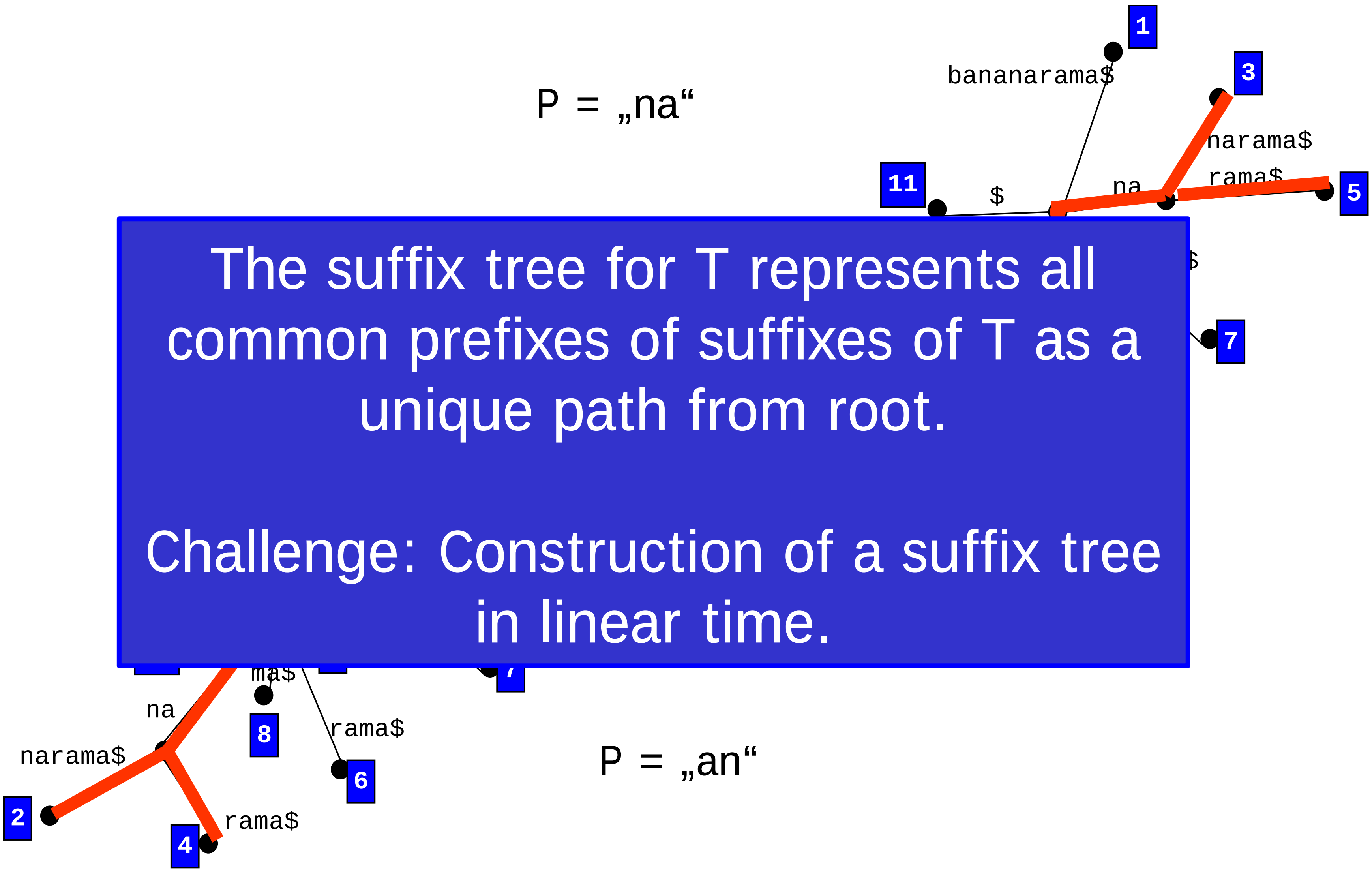
- Recall the problem to find all occurrences of a (short) string P in a (long) string T
- Fastest way (O(|P|)): Suffix Trees
  - Loot at all suffixes of T (there are |T| many)
  - Construct a tree
    - Every edge is labeled with a letter from T
    - All edges emitting from a node are labeled differently
    - Every path from root to a leaf is uniquely labeled
    - All suffixes of T are represented as leaves
- Every occurrence of P must be the prefix of a suffix of T
- Thus, every occurrence of P must map to a path starting at the root of the suffix tree

# Example

```
12345678901
BANANARAMA$
 ANANARAMA$
  NANARAMA$
   ANARAMA$
    NARAMA$
     ARAMA$
      RAMA$
       AMA$
        MA$
         A$
          $
```



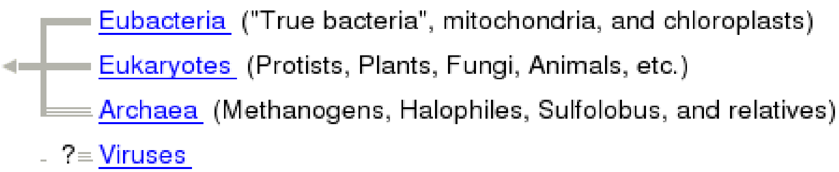
# Searching in the Suffix Tree




The suffix tree for T represents all common prefixes of suffixes of T as a unique path from root.

Challenge: Construction of a suffix tree in linear time.

# Classifications – Phylogenetic Trees

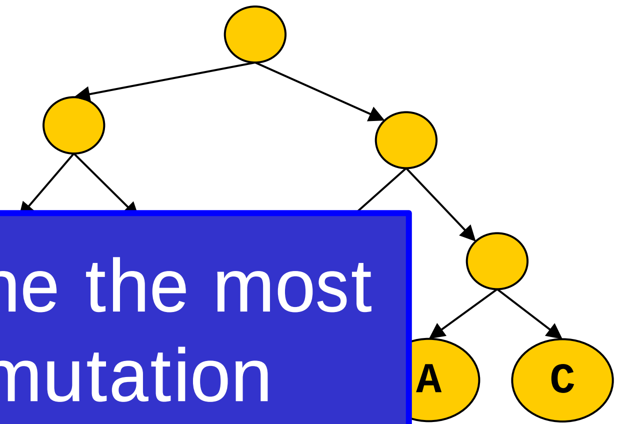



- Eukaryoten
- Tiere
- diverse Zwischenstufen
- Chraniata (Schädelknochen)
- Vertebraten (Wirbeltier)
- Viele Zwischenstufen
- Mammals (Säugetiere)
- Eutheria (Placenta)
- Primaten (Affen)
- Catarrhini
- Hominidae (Mensch, Schimpanse, Orang-Utan, Gorilla)
- Homo (erectus, sapiens …)
- Homo Sapiens

**Popular Groups on the Tree of Life**

- Eubacteria
- **Eukaryotes**
  - **Animals**
    - Echinoderms (sea urchins, starfish, sea cucumbers, etc)
    - **Vertebrates (fish etc.)**
      - **Terrestrial Vertebrates**
        - Frogs
        - Salamanders
        - Turtles
        - **Dinosaurs**
          - Modern Birds
        - Mammals
      - Teleost fish
    - Cnidaria (jellyfish, anemones, corals, etc.)
    - Annelida (segmented worms)
    - Cephalopoda (octopods, squids, etc.)
    - **Arthropoda**
      - **Insects**
        - Dragonflies and Damselflies
        - Lice
        - True Bugs
        - Beetles
        - Wasps, Bees, and Ants
        - Flies
        - Butterflies and Moths
        - Crickets, Katydids, and Grasshoppers
      - **Arachnids**
        - Spiders
        - Mites
        - Scorpions
  - Fungi
  - **Green Plants**
    - Ferns
    - Flowering Plants

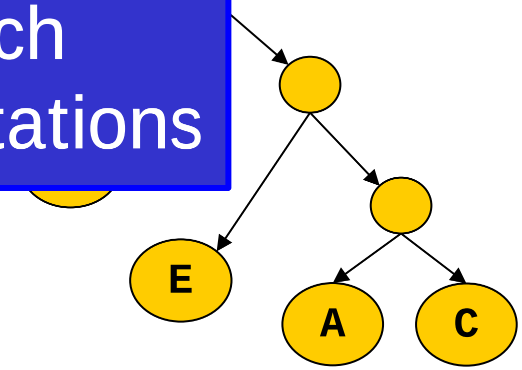
# Tree Construction




A: A
B: _
C: A
D: A
E: _

Phylogenetic Trees determine the most likely temporal order of mutation events.

Challenge: Find the tree which assumes the least number of mutations



# Graphs

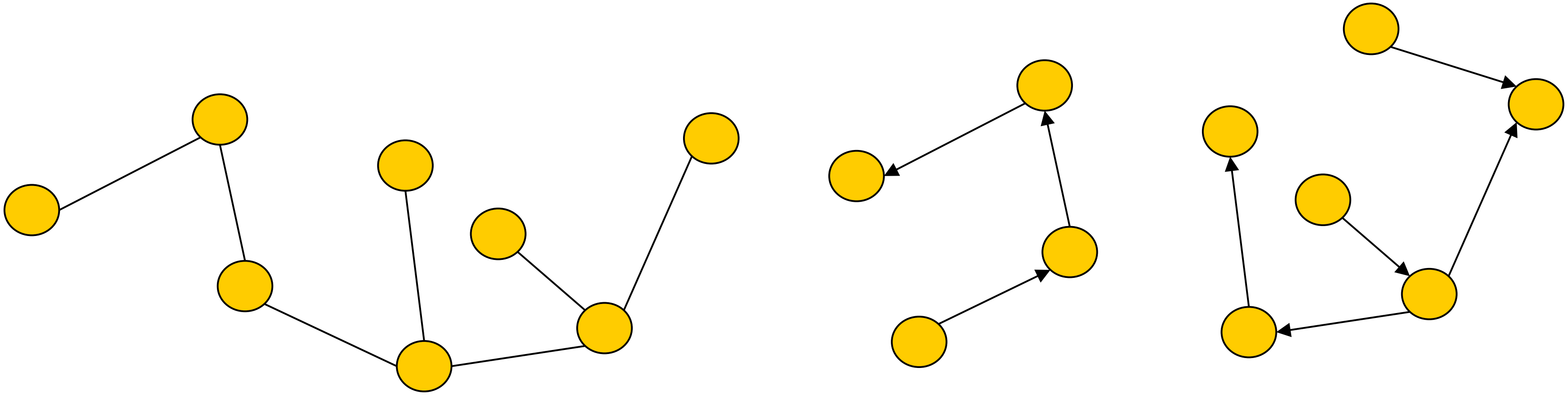
- Definition

  *A graph $G=(V, E)$ consists of a set V of vertices (nodes) and a set E of edges ($E \subseteq VxV$).*
  - *A sequence of edges $e_1, e_2, .., e_n$ is called a path iff $\forall 1 \leq i<n-1$: $e_i=(v', v)$ and $e_{i+1}=(v, v'')$*
  - *The length of a path $e_1, e_2, .., e_n$ is n*
  - *A path $(v_1,v_2), (v_2,v_3), ..., (v_{n-1},v_n)$ is acyclic iff all $v_i$ are different*
  - *G is connected if every pair $v_i, v_j$ is connected by at least one path*
  - *G is undirected, if $\forall(v,v')\in E \Rightarrow (v',v)\in E$. Otherwise G is directed*
  - *G is acyclic if it contains no cyclic path*

  *Let $G=(V, E)$ be a directed graph and let $v,v'\in V$.*
  - *Every edge $(v,v')\in E$ is called outgoing for v*
  - *Every edge $(v',v)\in E$ is called incoming for v*

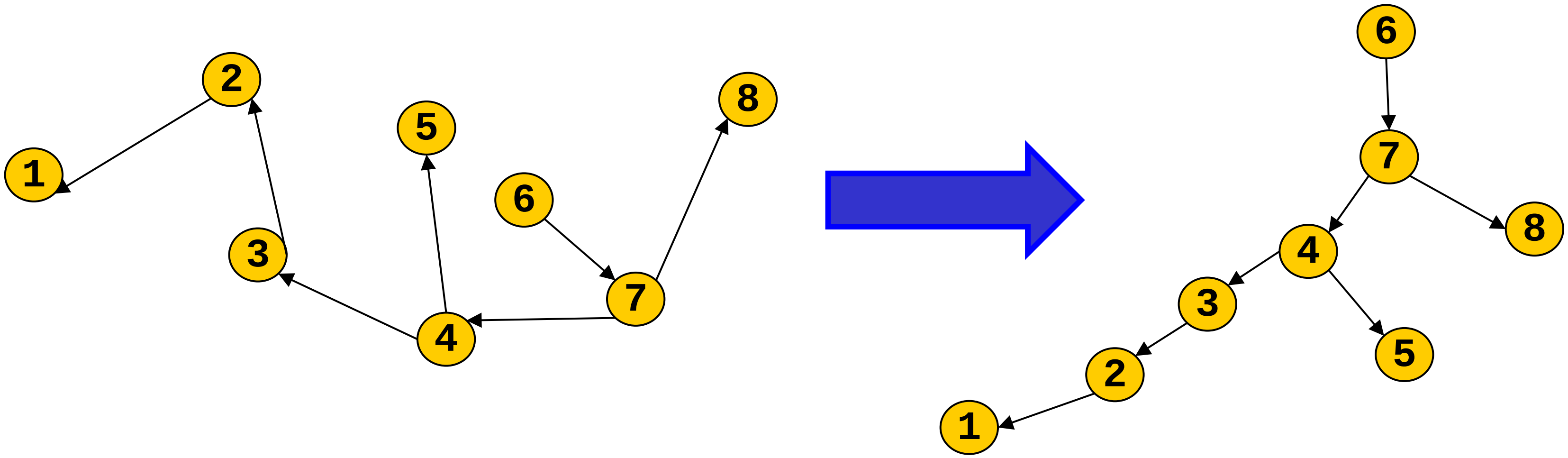
# Trees as Connected Graphs

- Definition
  - *A undirected connected acyclic graph is called a undirected tree*
  - *A directed connected acyclic graph in which every node has at most one incoming edge is called a directed tree*
- Lemma
  - *In a undirected tree, there exists exactly one path between any pair of nodes*

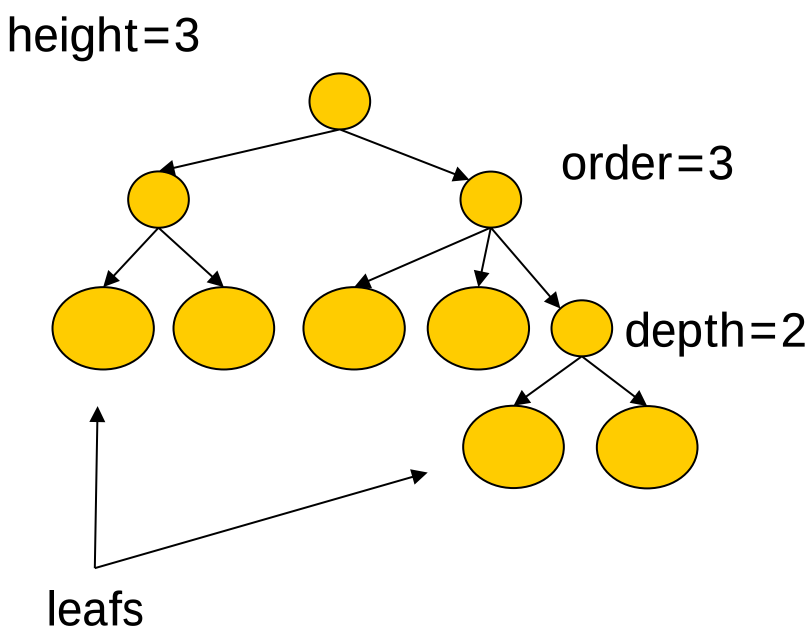


# Rooted Trees

- Definition
  *A directed tree with exactly one vertex v with no incoming edge is called a rooted tree; v is called the root of the tree*
- From now on: “Tree” means “rooted tree”
- Lemma
  - *In a rooted tree, there exists exactly one path between root and any other node*

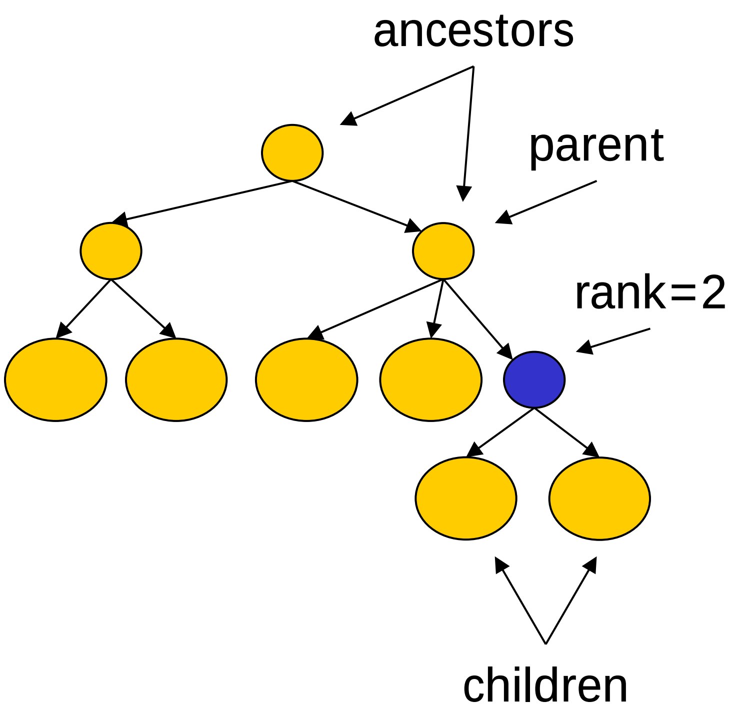


# Terminology



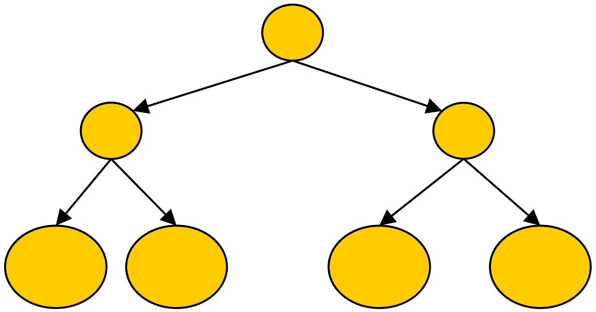

- Definition
  *Let T be a tree. Then …*
  - *A node with no outgoing edge is a leaf; other nodes are inner nodes*
  - *The depth of a node p is the length of the (only) path from root to p*
  - *The height of T is the depth of its deepest leaf*
  - *The order of T is the maximal number of children of its nodes*
  - *"Level i" are all nodes at depth i*
  - *T is ordered if the children of all inner nodes are ordered*

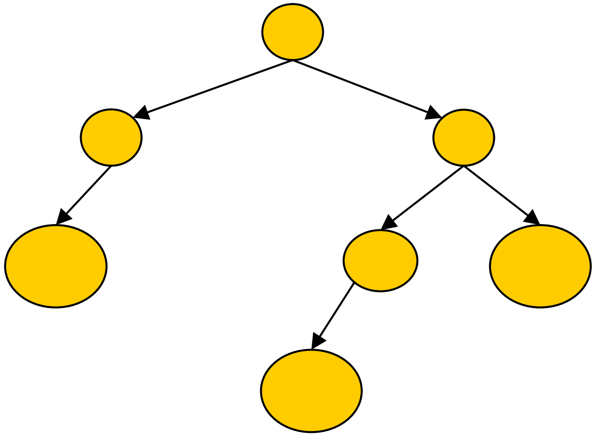
# More Terminology




- Definition
  *Let T be a tree and v a node of T. Then ...*
  - *All nodes adjacent to an outgoing edge of v are its children*
  - *v is called the parent of all its children*
  - *All nodes on the path from root to v are the ancestors of v*
  - *All nodes reachable from v are its successors*
  - *The rank of a node v is the number of its children*

# Two More Concepts





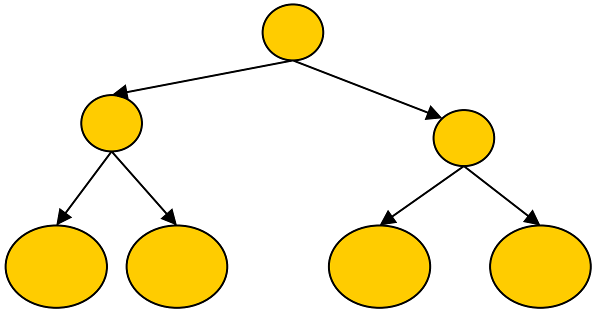
- Definition
  *Let T be a directed tree of order k. T is complete if all its inner nodes have rank k and all leaves have the same depth*

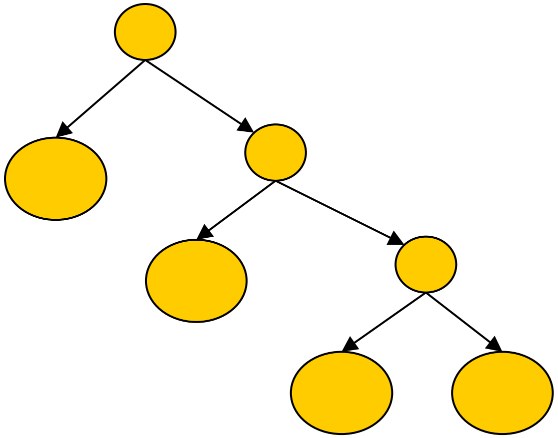
- In this lecture, we will mostly consider rooted ordered trees of order two (binary trees)

# Recursive Definition of Trees

- We defined trees as graphs with certain constraints
- Will mostly traverse trees using recursive functions
- The relationship may become clearer when using a recursive definition of (binary) trees
- Definition
  *A tree is a structure defined as follows:*
  - *A single node and an empty set is a tree with height 0*
  - *If $T_1$ and $T_2$ are (possible empty) trees, then the structure formed by a new node v and edges from v to the root of $T_1$ and from v to the root of $T_2$ is a tree*
    - *v is its root*
    - *The height of this tree is max(height($T_1$), height($T_2$))+1;*

# Some Properties (without proofs)





- Lemma

  *Let T=(V, E) be a tree of order k. Then*

  - $|V|=|E|+1$
  - *If T is complete, T has* $k^{height(T)}$ *leaves*
  - *If T is a complete binary tree, T has* $2^{height(T)+1}-1$ *nodes*
  - *If T is a binary tree with n leaves,* $height(T) \in [floor(log(n)), n-1]$

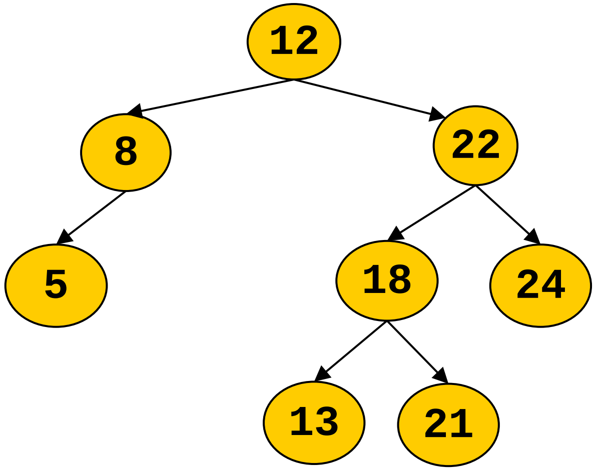
# Content of this Lecture

- Trees
- Search Trees
  - Definition
  - Searching
  - Inserting
  - Deleting
- Natural Trees

# Search Trees

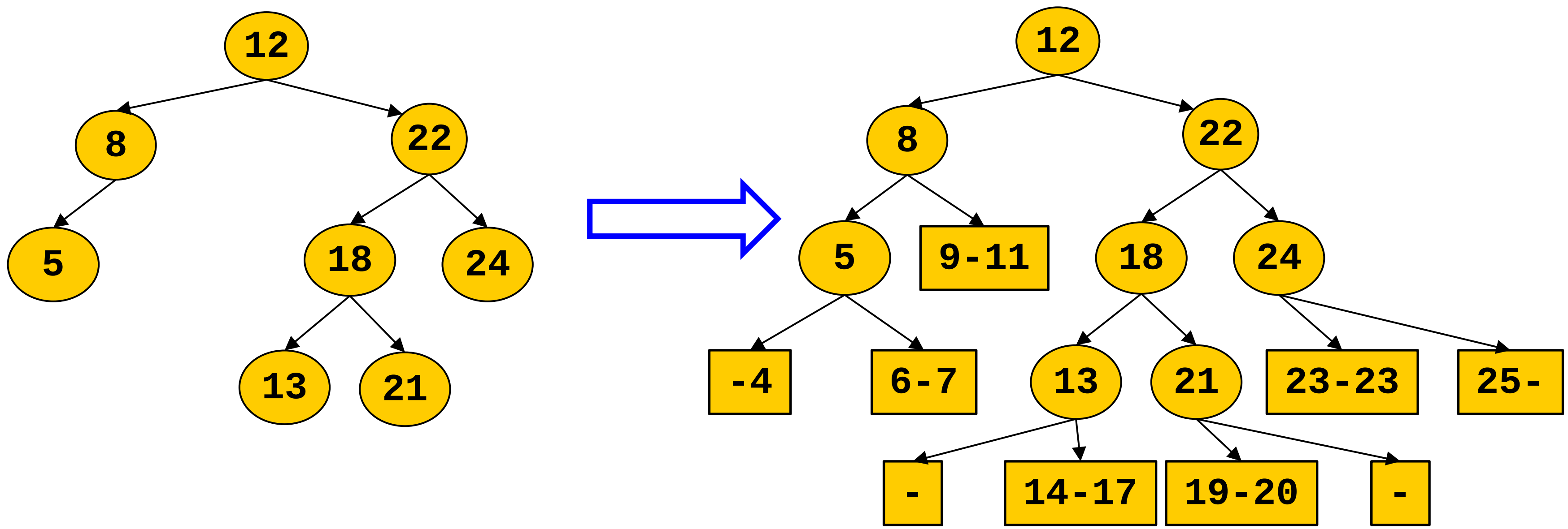
- Definition

  *A search tree T=(V,E) is a rooted binary tree with n=|V| differently key-labeled nodes such that ∀v∈V:*
  - *label(v)>max(label(left_child(v)), label(successors(left_child(v)))*
  - *label(v)<min(label(right_child(v)), label(successors(right_child(v)))*
- Remarks
  - For simplicity, we use integer labels
  - "node" ~ "label of a node"
  - We only consider search trees without duplicate keys (easy to change)
  - Search trees are used to manage and search a list of keys
  - Operations: search, insert, delete



# Complete Trees

- Conceptually, we pad search trees to full rank in all nodes
  - “padded” leaves are usually neither drawn nor implemented (NULL)
- A “padded” leaf represents the interval of values that would be below this node



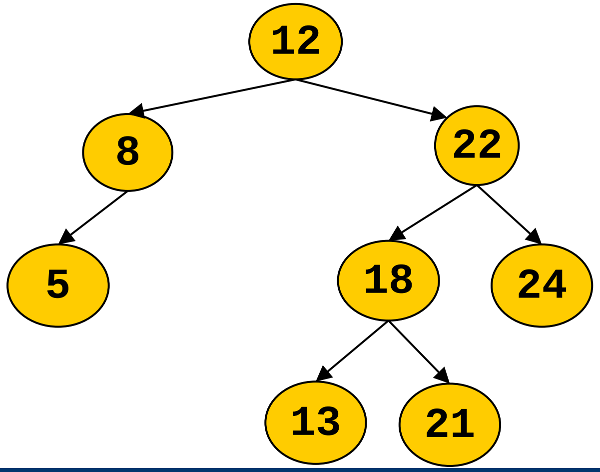
# What For?

- For a search tree T=(V,E), we eventually will reach O(log(|V|)) for testing whether k∈T and for inserting and deleting a key
  - First: Average Case of natural trees
  - Next: Worst Case for AVL-Trees
- Compared to binsearch on arrays, search trees are a dynamically growing / shrinking data structure
  - But need to store pointers

# Searching

- Searching a key k
  - Comparing k to a node determines whether we have to look further down the left or the right subtree
    - We stop if label(node)=k
  - If there is no child left, k∉T
- Complexity
  - In the worst case we need to traverse the longest path in T to show k∉T
  - Thus: O(|V|)
  - Wait a bit ...

```
func node search( T search_tree,
                  k integer) {
  v := root(T);
  while v!=null do
    if label(v)>k then
      v := v.left_child();
    else if label(v)<k then
      v := v.right_child();
    else
      return v;
  end while;
  return null;
}
```

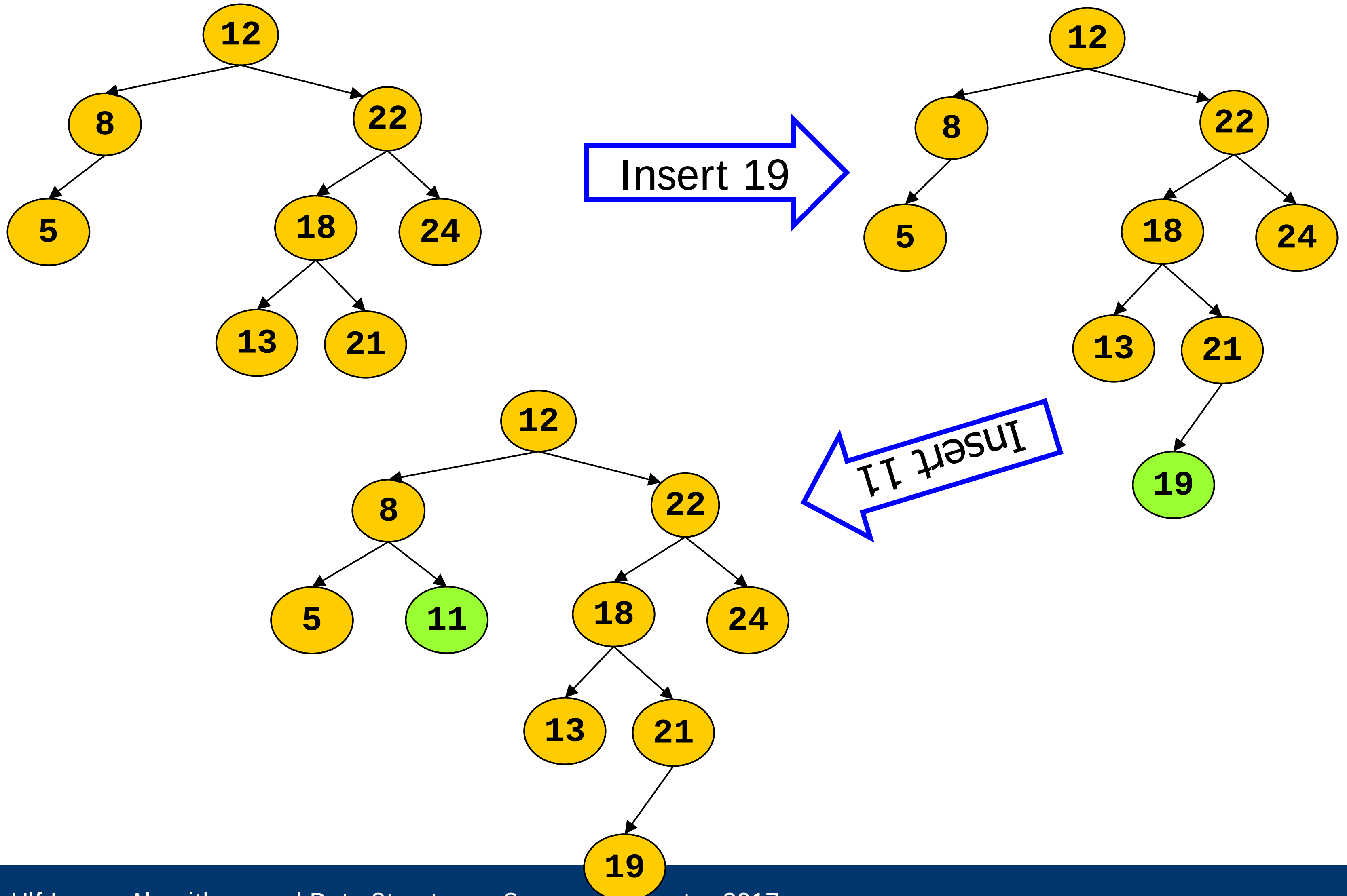


# Insertion

```
func bool insert( T search_tree,
                  k integer) {
  v := root(T);
  while v!=null do
    p := v;
    if label(v)>k then
      v := v.left_child();
    else if label(v)<k then
      v := v.right_child();
    else
      return false;
  end while;
  if label(p)>k then
    p.left_child := new node(k);
  else
    p.right_child := new node(k);
  end if;
  return true;
}
```

- First search the new key k
  - If $k \in T$, we do nothing
  - If $k \notin T$, the search must finish at a null pointer in a node p
    - A “right pointer” if label(p)<k, otherwise a “left pointer”
- We replace the null with a pointer to a new node k
- Complexity: Same as search

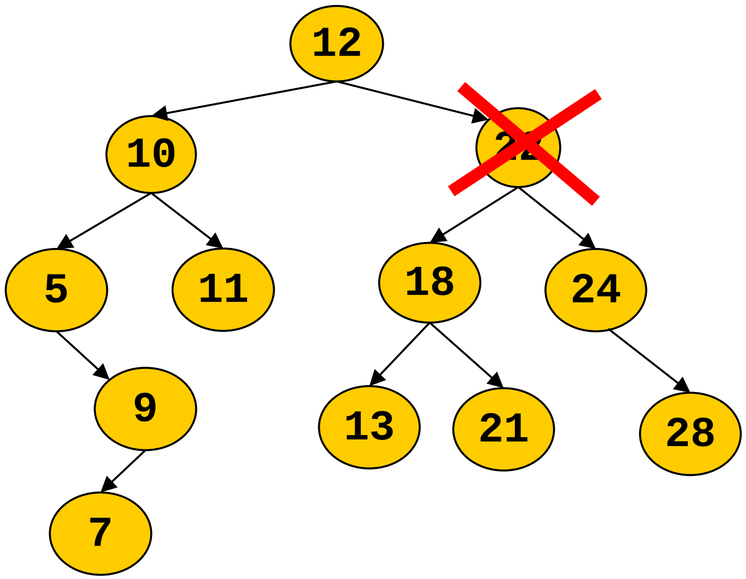
# Example


12
8
22
5
18
24
13
21
Insert 19
12
8
22
5
18
24
13
21
19
Insert 11
12
8
22
5
11
18
24
13
21
19

# Deletion

- Again, we first search k
- If $k \notin T$, we are done
- Assume $k \in T$. The following situations are possible
  - k is stored in a leaf. Then simply remove this leaf
  - k is stored in an inner node q with only one child. Then remove q and connect parent(q) to child(q)
  - k is stored in an inner node q with two children. Then ...

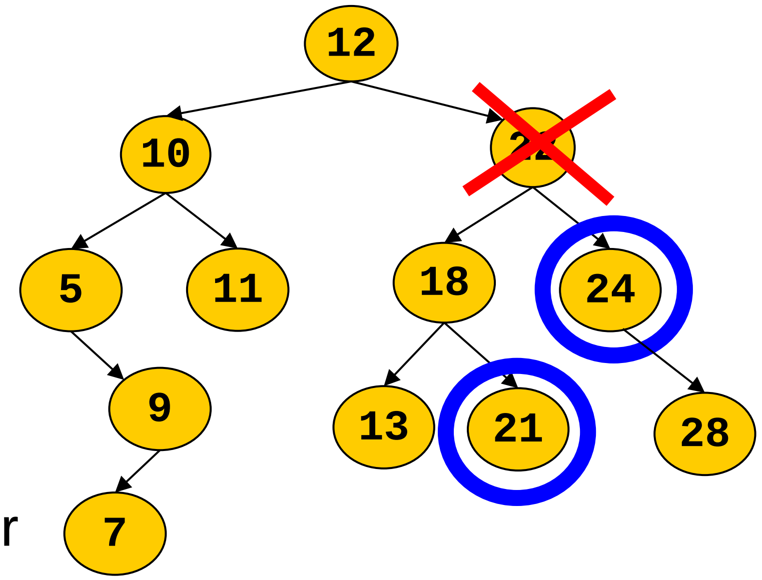
# Observations




- We cannot remove q, but we can replace the label of q with another label - and remove this node
- We need a node q’ which can be removed and whose label k’ can replace k without hurting the search tree constraints
  - label(k’)>max(label(left_child(k’)), label(successors(left_child(k’)))
  - label(k’)<min(label(right_child(k’)), label(successors(right_child(k’)))

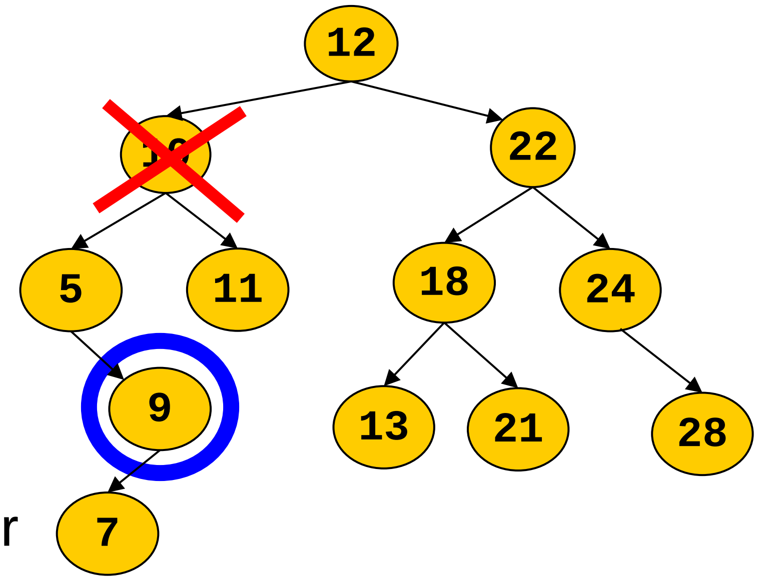
# Observations

- Two candidates
  - Largest value in the left subtree (symmetric predecessor of k)
  - Smallest value in the right subtree (symmetric successor of k)
- We can choose any of those
  - Let's use the symmetric predecessor
  - This is either a leaf – no problem

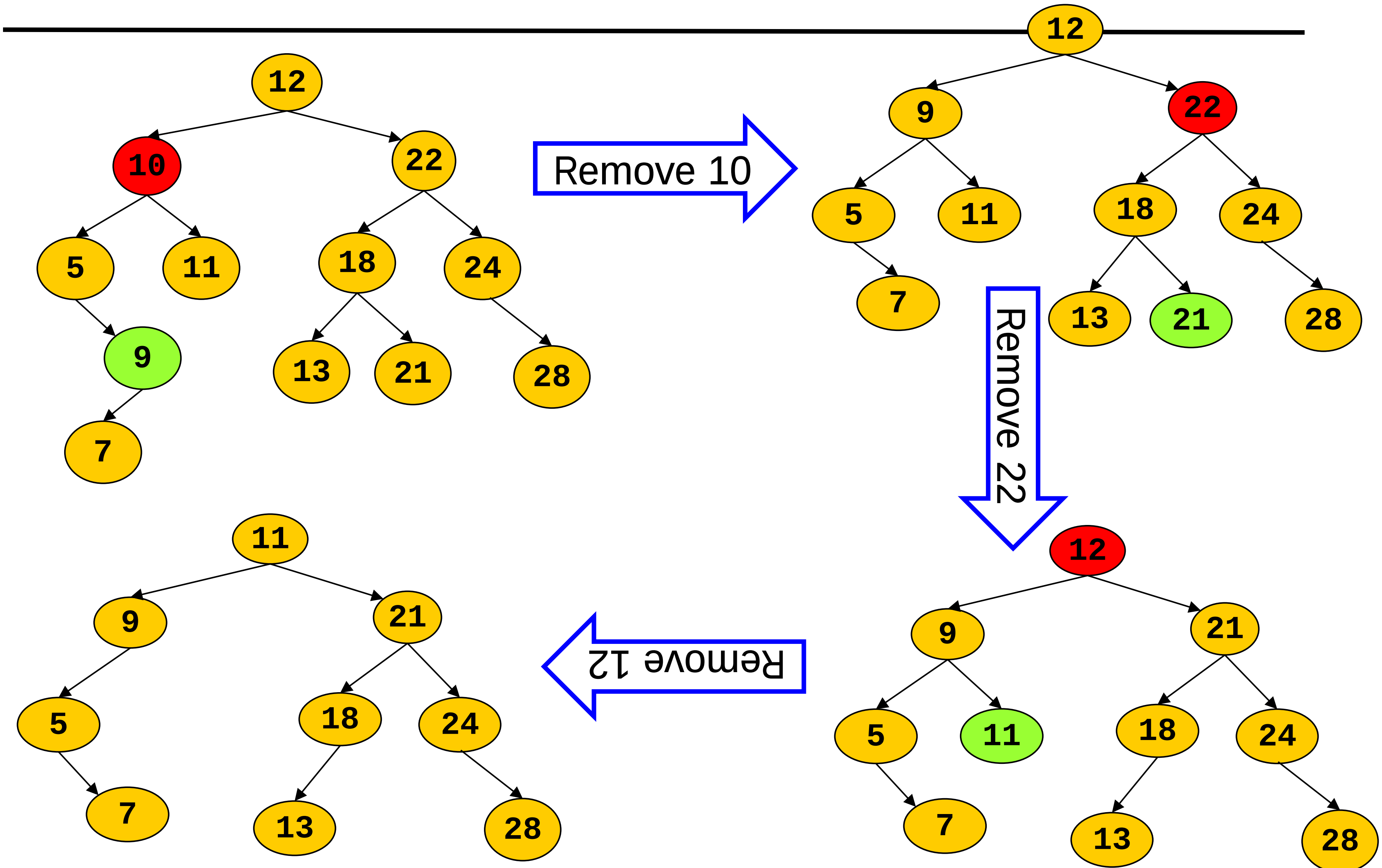


# Observations

- Two candidates
  - Largest value in the left subtree (symmetric predecessor of k)
  - Smallest value in the right subtree (symmetric successor of k)
- We can choose any of those
  - Let’s use the symmetric predecessor
  - This is either a leaf
  - Or an inner node; but since its label is larger than that of all other labels in the left subtree of q, it can only have a left child
  - Thus it is a node with one child - and can be removed easily



# Example



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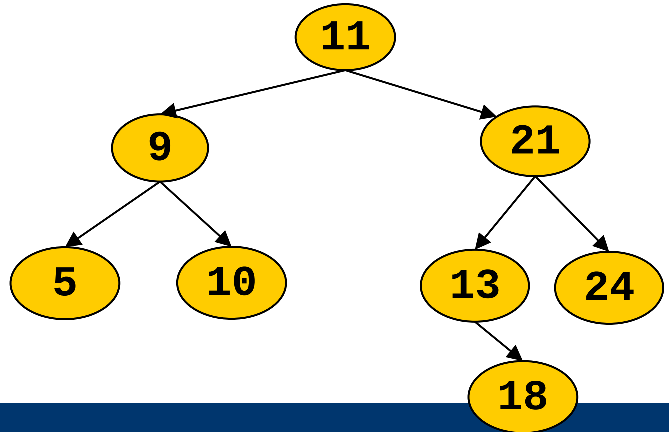
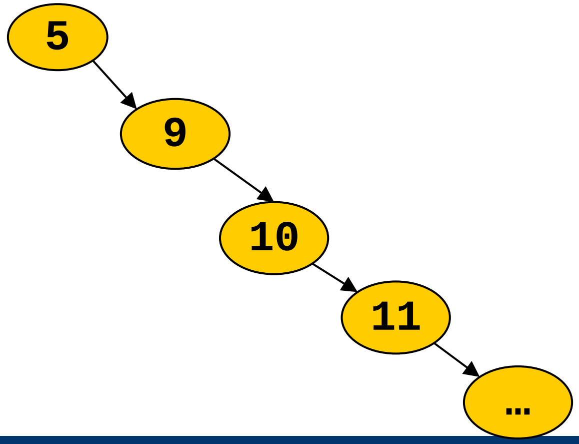
# Natural Trees

- A search tree created by inserting and deleting keys in a random order is called a natural tree
- A natural tree T=(V,E) has height(T)∈[|V|-1, log(|V|)]
- The concrete height for a set of keys depends on the order in which keys were inserted
- Example

11,9,10,5,21,13,24,18



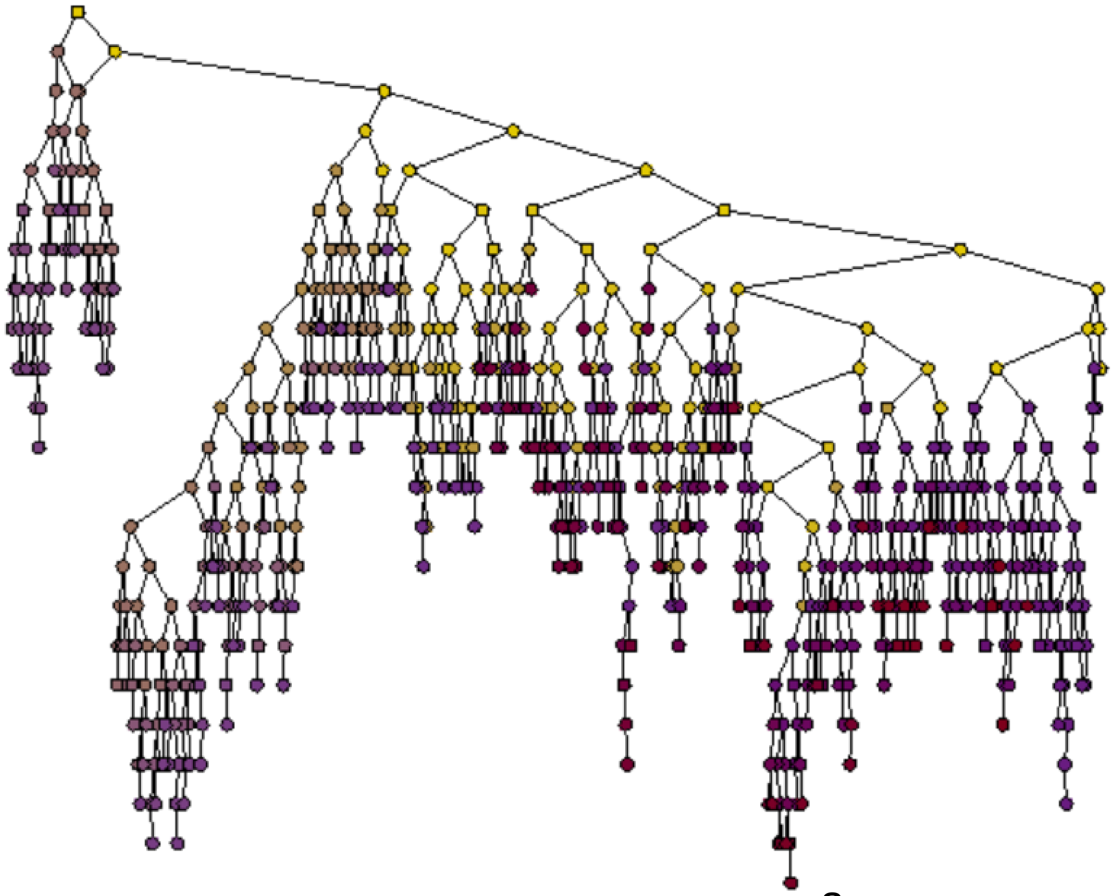

5,9,10,11,13,18,21,24



# Average Case Analysis

- A natural tree with n nodes has maximal height of n-1
- Thus, searching will need O(n) comparisons in worst-case
  - Same for inserting and deleting
- But: Natural trees are not bad on average
  - The average case is O(log(n))
  - More precisely, a natural tree is on average only ~1.4 times deeper than the optimal search tree (with height h~log(n))
  - We skip the proof (argue over all possible orders of inserting n keys), because balanced search trees (AVL trees) are O(log(n)) also in worst-case and are not much harder to implement

# Example



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# Exemplary Questions

- Construct a natural search tree from the following input, showing all intermediate steps (I: insert; D: delete): I5, I7, I3, I10, D7, I7, I13, I12, D5
- The worst case complexity for inserting/deleting a key into a search tree with n=|V| nodes is O(n). Give an order of the following operations such that this worst case happens for every operation: I5, I7, I3, I10, D7, I7, I13, I12, D5
- For deleting a given key k in a natural search tree, one may need to find the symmetric predecessor (SP) of a key. Define what a SP is, give an algorithm for finding it (starting from k), and analyze its complexity